

Abelian Varieties with Complex Multiplication

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An **abelian variety** over a field k is a smooth, geometrically integral, projective group scheme A over k . In other words, it is a projective scheme A equipped with the addition operation $+$: $A \times A \rightarrow A$ and the inverse map i : $A \rightarrow A$ that are both morphisms of varieties. The projectiveness of A forces the group operation to be commutative (see [Mil08, Cor 1.4]).

Typically, the endomorphism ring of an abelian variety over $\mathbb{C}$ with dimension g is $\mathbb{Z}$ (with the multiplication-by- n map being the only endomorphisms). However, in the extreme case, it can be an order of a number field of degree $2g$. The goal of these notes is to develop the theory about such abelian varieties.

§1 Abelian Varieties as a Complex Torus

Let A be an abelian variety over $\mathbb{C}$ of dimension n . Note that A is a complex Lie group. Thus, we have the exponential map

$$\exp : \mathbb{C}^n \simeq T_e A \rightarrow A,$$

which one can show that it is a covering map. The kernel of this map is a discrete subgroup of $\mathbb{C}^n$, so it is a $2n$ -dimensional lattice Λ . In particular, we deduce that

$$A \simeq \mathbb{C}^n / \Lambda \text{ for some lattice } \Lambda \subseteq \mathbb{C}^n$$

(isomorphic means as complex manifolds). In other words, an abelian variety with dimension n is holomorphic to a complex n -torus.

Suppose we have two abelian varieties A and B of dimension n . Thus $A \simeq \mathbb{C}^n / \Lambda_A$ and $B \simeq \mathbb{C}^n / \Lambda_B$ for some lattices Λ_A and Λ_B . Then any homomorphism $A \rightarrow B$ (by that we mean maps $A \rightarrow B$ that preserves group operations) correspond to a holomorphic map on $\mathbb{C}^n / \Lambda_A \rightarrow \mathbb{C}^n / \Lambda_B$. It is not hard to check that a biholomorphic map between two complex tori are linear. In particular, we deduce that

$$\{\text{homomorphisms } A \rightarrow B\} \longleftrightarrow \{\mathbb{C}\text{-linear maps } \mathbb{C}^n \rightarrow \mathbb{C}^n \text{ that sends } \Lambda_A \rightarrow \Lambda_B\}.$$

A morphism $A \rightarrow B$ is an **isogeny** if it has a finite kernel, i.e., it is a $\mathbb{C}$ -linear map that induces an isomorphism in $\Lambda_A \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow \Lambda_B \otimes_{\mathbb{Z}} \mathbb{Q}$. Two abelian varieties A and B are **isogenous** if there exists an isogeny $A \rightarrow B$, which is equivalent to that $\Lambda_A \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \Lambda_B \otimes_{\mathbb{Z}} \mathbb{Q}$, so this is a symmetric relation.

✓ **Example 1** (Complex torus with endomorphism ring $\mathbb{Z}[\zeta_5]$).

Let $\zeta_5 = \exp(2\pi i/5)$. Consider the quotient of $\mathbb{C}^2$ by the lattice

$$\Lambda = \text{Span}\{(\zeta_5, \zeta_5^2), (\zeta_5^2, \zeta_5^4), (\zeta_5^3, \zeta_5), (\zeta_5^4, \zeta_5^3)\}.$$

Then the linear map $(x, y) \mapsto (\zeta_5 x, \zeta_5^2 y)$ is an endomorphism of $\mathbb{C}^2 / \Lambda$. In particular $\text{End}^0 \mathbb{C}^2 / \Lambda$ contains $\mathbb{Z}[\zeta_5]$, where the element ζ_5 correspond to the map in the previous sentence.

The previous example shows how higher-dimensional complex torus can have endomorphism ring be a number field of degree larger than 2.

Example 2.

More generally, let $E/\mathbb{Q}$ be a CM field of degree $2n$. Thus, there are $2n$ embeddings $E \hookrightarrow \mathbb{C}$ coming in n conjugate pairs. Pick a representative from each pair, giving embeddings $\sigma_1, \dots, \sigma_n$. Let I be an ideal in $\mathcal{O}_E$. Consider the complex torus A obtained by taking quotient of $\mathbb{C}^n$ with the lattice

$$\Lambda = \text{Span}\{(\sigma_1(\alpha), \dots, \sigma_n(\alpha)) : \alpha \in I\},$$

which has dimension n because the image after tensoring with $\mathbb{C}$ is the span of $\sigma_1(\alpha), \overline{\sigma_1(\alpha)}, \dots, \sigma_n(\alpha), \overline{\sigma_n(\alpha)}$.

The endomorphism ring $\text{End } A$ contains the map $(x_1, \dots, x_n) \mapsto (\sigma_1(\alpha)x_1, \dots, \sigma_n(\alpha)x_n)$ for each $\alpha \in \mathcal{O}_E$. Thus, $\text{End}^0 A$ contains E (in fact, it is equal to E).

The set of embedding $\Phi = \{\sigma_1, \dots, \sigma_n\}$ is called a **CM type**.

§2 Riemann Form

Are all complex tori abelian varieties? Unfortunately, the answer is no.

The reason for this is we need an ample line bundle $\mathcal{L}$ on the complex torus $\mathbb{C}^n/\Lambda$. This ample line bundle has the **Chern class** $c_1(\mathcal{L}) \in H^2(\mathbb{C}^n/\Lambda, \mathbb{Z})$, coming from the boundary map

$$H^1(\mathbb{C}^n/\Lambda, \mathcal{O}^\times) \rightarrow H^2(\mathbb{C}^n/\Lambda, \mathbb{Z})$$

in the exact sequence

$$0 \rightarrow \mathbb{Z} \rightarrow \mathcal{O} \xrightarrow{f \mapsto \exp(2\pi i f)} \mathcal{O}^\times \rightarrow 0.$$

Since $H^2(\mathbb{C}^n/\Lambda, \mathbb{Z}) \simeq \bigwedge^2 H^1(\mathbb{C}^n/\Lambda, \mathbb{Z})$, the Chern class corresponds to an alternating pairing

$$\Lambda \times \Lambda \simeq H_1(\mathbb{C}^n/\Lambda, \mathbb{Z}) \times H_1(\mathbb{C}^n/\Lambda, \mathbb{Z}) \rightarrow \mathbb{Z},$$

and tensoring with $\mathbb{R}$ gives an $\mathbb{R}$ -linear alternating pairing

$$\mathbb{C}^n \times \mathbb{C}^n \simeq (\Lambda \otimes_{\mathbb{Z}} \mathbb{R}) \times (\Lambda \otimes_{\mathbb{Z}} \mathbb{R}) \rightarrow \mathbb{R}$$

that sends $\Lambda \times \Lambda \rightarrow \mathbb{Z}$. This alternating pairing gives the Riemann form below, where the complex structure of Chern class gives (c) and the ampleness gives (d).

Theorem 3.

A complex torus $\mathbb{C}^n/\Lambda$ is an abelian variety if and only if there exists a **Riemann form**, which is a $\mathbb{R}$ -linear pairing $\psi : \mathbb{C}^n \times \mathbb{C}^n \rightarrow \mathbb{R}$ such that

- (a) $\psi(\alpha, \beta) \in \mathbb{Z}$ for all $\alpha, \beta \in \Lambda$.
- (b) ψ is alternating: $\psi(v, w) = -\psi(w, v)$ for all $v, w \in \Lambda_{\mathbb{R}}$;
- (c) $\psi(iv, iw) = \psi(v, w)$ for all $v, w \in \Lambda_{\mathbb{R}}$;
- (d) $\psi(v, iv) > 0$ for all nonzero $v \in \Lambda_{\mathbb{R}}$.

(Here, multiplication by i is done coordinate-wise.)

Note that in the first condition, it suffices that $\psi(\alpha, \beta) \in \mathbb{Q}$, and then we can scale ψ by a positive integer amount.

Example 4.

The quotient of $\mathbb{C}^2$ by the lattice

$$\Lambda = \text{Span}_{\mathbb{Z}}\{(1, 0), (i, 0), (\sqrt{10}, i), (0, 1)\}$$

does not admit the Riemann form. To see why, let $e_1 = (1, 0)$ and $e_2 = (0, 1)$. Then

$$\begin{aligned} \underbrace{\psi(\sqrt{10}e_1 + ie_2, ie_1)}_{\in \mathbb{Z}} &= \sqrt{10}\psi(e_1, ie_1) + \psi(ie_2, ie_1) \\ &= \sqrt{10}\underbrace{\psi(e_1, ie_1)}_{\in \mathbb{Z}} + \underbrace{\psi(e_2, e_1)}_{\in \mathbb{Z}}, \end{aligned}$$

so $\psi(e_1, ie_1) = 0$, contradicting positive-definiteness.

The above example shows that a random complex torus is unlikely to be an abelian variety. We now show that the complex torus in [Example 2](#) is indeed an abelian variety.

Proposition 5.

Let $E/\mathbb{Q}$ be a CM field of degree $2n$ and $\Phi = \{\sigma_1, \dots, \sigma_n\}$ be a CM type. Let I be a fractional ideal in E . Then the complex torus A obtained by taking quotient of $\mathbb{C}^n$ with the lattice

$$\Lambda = \{(\sigma_1(\alpha), \dots, \sigma_n(\alpha)) : \alpha \in I\}.$$

is an abelian variety.

► **Proof.** Let $P_\alpha = (\sigma_1(\alpha), \dots, \sigma_n(\alpha))$.

We claim that this admits a Riemann form given by

$$\psi(x, y) = \sum_{k=1}^n \text{Re}(\sigma_k(t)x_k \overline{y_k})$$

for some $t \in E$. We note that

- for any $\alpha, \beta \in E$, we have

$$\psi(P_\alpha, P_\beta) = \sum_{k=1}^n \text{Re}(\sigma_k(t\alpha\bar{\beta})) = \frac{1}{2} \sum_{k=1}^n (\sigma_k(t\alpha\bar{\beta}) + \overline{\sigma_k(t\alpha\bar{\beta})}) = \frac{1}{2} \text{Tr}_{E/\mathbb{Q}}(t\alpha\bar{\beta}) \in \mathbb{Q},$$

so condition (a) is automatic.

- if t is purely imaginary, then $\sigma_k(t)$ is also purely imaginary, so

$$\psi(x, y) = \frac{1}{2} \sum_{k=1}^n \sigma_k(t)(x_k \overline{y_k} - \overline{x_k} y_k) = -\psi(y, x),$$

so condition (b) is satisfied.

- the condition $\psi(x, y) = \psi(ix, iy)$ holds regardless of t ;
- we have

$$\psi(x, ix) = \sum_{k=1}^n \operatorname{Re}(-i\sigma_k(t)|x_k|^2) = \sum_{k=1}^n \operatorname{Im}(\sigma_k(t))|x_k|^2$$

which is positive if $\operatorname{Im}(\sigma_k(t)) > 0$ for all $k = 1, 2, \dots, n$.

Thus, we have to find $t \in E$ purely imaginary such that $\sigma_k(\operatorname{Im} t) > 0$ for all $k = 1, 2, \dots, n$. This can be done by the weak approximation theorem by choosing u to be close to i at places $\sigma_1, \dots, \sigma_k$ and then take $t = \frac{u - \bar{u}}{2}$. $\square$

Remark 6.

One can check that all Riemann forms must be of the above form, up to scaling.

§3 Poincaré Reducibility Theorem

Abelian varieties can be uniquely decomposed to product of abelian varieties as follows.

Theorem 7 (Poincaré Reducibility Theorem).

Every abelian variety A is isogenous to a product

$$B_1^{n_1} \times \cdots \times B_k^{n_k}$$

where B_i are **simple** (i.e., no nontrivial abelian subvariety) and no two of B_i and B_j are isogenous. Moreover, this decomposition is unique up to permuting factors and replacing B_i 's with isogenous varieties.

► **Proof Sketch.** It suffices to show that if B is a subvariety of A , then there exists an abelian variety B' such that A is isogenous to $B \times B'$. This is done by taking orthogonal complement with respect to the Riemann form. $\square$

We now use this to study endomorphism rings of abelian varieties. Let $\operatorname{End}^0 A = \operatorname{End} A \otimes_{\mathbb{Z}} \mathbb{Q}$.

Proposition 8.

Let A be an abelian variety over $\mathbb{C}$.

- If A is simple, then $\operatorname{End}^0 A$ is a division algebra.
- $\operatorname{End}^0 A$ is semisimple.

► **Proof.** (a) This follows from “Schur’s lemma”. If the endomorphism has a nontrivial kernel/image (when tensored with $\mathbb{Q}$), then there would be a nontrivial subvariety. Thus, every nonzero endomorphism is invertible.

(b) Note that if A is isogenous to a product

$$B_1^{n_1} \times \cdots \times B_k^{n_k},$$

where no two of $B_1, \dots, B_k$ are isogenous and simple. Then

$$\mathrm{End}^0 A = \prod_{j=1}^k \mathrm{Mat}_{n_j \times n_j}(\mathrm{End}^0 B_j),$$

since $\mathrm{Hom}(B_i, B_j) = 0$ for non-isogenous simple varieties B_i and B_j by (a). Thus, $\mathrm{End}^0 A$ is semisimple. $\square$

§4 Abelian Varieties with Complex Multiplication

We recall the following result on simple algebras.

Lemma 9.

Let B be a semisimple k -algebra. For any faithful B -module M , we have

$$\dim_k M \geq [B : k]_{\mathrm{red}}.$$

Moreover, if the equality holds, then B is a product of matrix algebras over fields.

(We are not going to define $[B : k]_{\mathrm{red}}$, but for example $[M_n(K) : k]_{\mathrm{red}} = n[K : k]$.)

► *Proof.* See [Mil20, Prop 1.2]. $\square$

Corollary 10.

For any abelian variety A over $\mathbb{C}$, we have that

$$[\mathrm{End}^0 A : \mathbb{Q}]_{\mathrm{red}} \leq 2 \dim A$$

Moreover, if the equality holds, then $\mathrm{End}^0 A$ is a product of matrix algebras over fields.

► *Proof.* Follows from above since $\mathrm{End}^0 A$ is a semisimple algebra (by Proposition 8) acting on $\Lambda_{\mathbb{Q}} \simeq \mathbb{Q}^{2g}$. $\square$

Definition 11.

An abelian variety A over $\mathbb{C}$ has a **complex multiplication** if and only if $[\mathrm{End}^0 A : \mathbb{Q}]_{\mathrm{red}} = 2 \dim A$.

Note that (cf. the proof of Proposition 8) A has a complex multiplication if and only if its simple factors have complex multiplication. Thus, we restrict our attention to understanding **simple** abelian varieties with CM.

Theorem 12 (Classification of all simple CM abelian varieties).

Let A be a simple abelian variety with dimension n . Then there exists a CM field E of degree $2n$ and a primitive CM type $\Phi = \{\sigma_1, \dots, \sigma_n\}$ such that $E = \mathrm{End}^0 A$ and $A \simeq \mathbb{C}^n / \Lambda$ where

$$\Lambda = \{(\sigma_1(\alpha), \dots, \sigma_n(\alpha)) : \alpha \in L\}$$

for some lattice $L \subseteq E$.

(Recall that a CM Type Φ on E is primitive if and only if E is a field and there does not exist a CM type (Φ_0, E_0) such that $E_0 \subsetneq E$ and $\Phi = \{\sigma : \sigma|_{E_0} \in \Phi_0\}$.)

► **Proof of Theorem 12.** The proof proceeds in five steps.

Step 1. $E = \text{End}^0 A$ is a field. From Proposition 8, E is a division algebra. From Corollary 10, E is a product of matrix algebras over fields. The two previous sentences give that E is a field itself.

We now let $A = \mathbb{C}^n / \Lambda$.

Step 2. $\Lambda = \{(\sigma_1(\alpha), \dots, \sigma_n(\alpha)) : \alpha \in L\}$ for some lattice $L \subseteq E$ and embeddings $\sigma_1, \dots, \sigma_n$. The endomorphism algebra E acts on $\mathbb{C}^n$. Since E is commutative and its minimal polynomial is separable, simultaneously diagonalize the matrix in $\mathbb{C}^{n \times n}$ corresponding to each $x \in E$. That is, there is a basis for which all such x is diagonal, meaning that x acts on $\mathbb{C}^n$ by the matrix

$$\text{diag}(\sigma_1(x), \dots, \sigma_n(x)).$$

Note that $\sigma_j(x)$ must be a field automorphism from $E \rightarrow \mathbb{C}$. Thus, it must be one of the $2n$ embeddings. Since Λ is stable under E , we have that

$$\Lambda = \{(\sigma_1(\alpha), \dots, \sigma_n(\alpha)) : \alpha \in L\}$$

for some lattice $L \subseteq E$, as desired.

Step 3. $\{\sigma_1, \dots, \sigma_n, \bar{\sigma}_1, \dots, \bar{\sigma}_n\} = \text{Hom}(E, \mathbb{C})$. The key here is that Λ must have dimension $2n$. This means that the $\mathbb{R}$ -span of $(\sigma_1(\alpha), \dots, \sigma_n(\alpha))$ as α ranges through E has dimension $2n$. Tensoring with $\mathbb{C}$ and using the isomorphism $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \rightarrow \mathbb{C} \oplus \mathbb{C}$ given by $a \otimes b \mapsto (ab, \bar{a}\bar{b})$ gives that

$$\dim_{\mathbb{C}}\{(\sigma_1(\alpha), \bar{\sigma}_1(\alpha), \dots, \sigma_n(\alpha), \bar{\sigma}_n(\alpha)) : \alpha \in E\} = 2n.$$

This is possible only if $\sigma_1, \bar{\sigma}_1, \dots, \sigma_n, \bar{\sigma}_n$ are pairwise distinct, which means they must be the $2n$ embeddings of E .

Step 4. E is CM. We have to use the Riemann form here. First, we define the form

$$\tilde{\psi}(x, y) = \psi(x, iy),$$

which is positive definite by the condition (d) of Theorem 3. Furthermore, it is symmetric because

$$\tilde{\psi}(y, x) = \psi(y, ix) = \psi(iy, -x) = \psi(x, iy) = \tilde{\psi}(x, y),$$

where we use condition (b) and (c) of Theorem 3.

Define the **Rosati involution** as a map from $\bullet' : E \rightarrow E$ that takes $\alpha \in E$ to $\alpha' \in E$ such that $\tilde{\psi}(\alpha x, y) = \tilde{\psi}(x, \alpha' y)$. (This requires the condition (a) on Theorem 3.) Since $\tilde{\psi}$ is symmetric and positive definite, we get that $\text{Tr}_{E/\mathbb{Q}}(\alpha \alpha') > 0$ for all nonzero $\alpha \in E$. Thus, $'$ is positive.

Tensoring with $\mathbb{R}$ gives an involution $\bullet' : E \otimes_{\mathbb{Q}} \mathbb{R} \rightarrow E \otimes_{\mathbb{Q}} \mathbb{R}$, which is also a ring automorphism. From Step 3, $E \otimes_{\mathbb{Q}} \mathbb{R} \simeq \mathbb{C}^{\oplus n}$, and so $\bullet'$ must permute the $\mathbb{C}$ factors. If $\bullet'$ swaps two of the factors, then it would not be positive since $\alpha \alpha' = 0$ if α is in one of the factors. Hence, $\bullet'$ fixes each factor and hence must be conjugation in each factor (the only positive field automorphism of $\mathbb{C}/\mathbb{R}$ is conjugation), so E is CM.

Step 5. Φ is primitive. Suppose that Φ comes from extending a CM type (Φ_0, E_0) . We consider the subspace

$$V = \{(x_1, \dots, x_n) \in \mathbb{C}^n : x_j = x_k \text{ for all } j, k \text{ such that } \sigma_j|_{E_0} = \sigma_k|_{E_0}\}.$$

This has dimension $[E_0 : \mathbb{Q}]$. Moreover, $V \cap \Lambda$ is isomorphic to the lattice $(\tau(\alpha))_{\tau \in \text{Hom}(E_0, \mathbb{C})}$ embedded diagonally. Thus, V is an abelian subvariety of A , which implies that A is not simple. $\square$

 Proposition 13 (Classification up to isogeny).

Let E be a CM field. Then there is a bijection between

$$\frac{\{\text{simple CM abelian variety } A \text{ with } \text{End}^0 A \simeq E\}}{\text{isogeny}} \simeq \text{Aut}(E) \setminus \{\text{primitive CM types of } E\},$$

where $\tau \in \text{Aut}(E)$ acts on CM type Φ by sending it to $\Phi\tau^{-1}$.

► *Proof.* First, note that the choice of I will not change the resulting abelian variety up to isogeny. Thus, the only choice remaining is to pick how E is embedded into $\text{End}^0 A$. This can be changed up to an automorphism $\tau \in \text{Aut } E$, and the effect of this on the CM type Φ is $\Phi\tau^{-1}$.

It remains to check that A obtained by the standard construction of $\Lambda = \{(\sigma_1(\alpha), \dots, \sigma_n(\alpha)) : \alpha \in L\}$ is simple if Φ is primitive. To do this, suppose that $A \simeq B_1^{n_1} \times \dots \times B_r^{n_r}$. Thus, $\text{End}^0(A) \simeq \prod_j \text{Mat}_{n_j}(\text{End}^0 B_j)$. Since E maps to each of the factors $\text{Mat}_{n_j}(\text{End}^0 B_j)$ injectively, we get that $\dim E \leq n_j [\text{End}^0 B_j : \mathbb{Q}]$ for all j . If $r > 1$, then this is strictly less than $[\text{End}^0 A : \mathbb{Q}]_{\text{red}}$, which is a contradiction to that A has a CM. Thus, we have $r = 1$, so $A \simeq B^n$ for some simple B .

Let $E_0 = \text{End}^0 B$. Then the restriction of $\sigma_1, \dots, \sigma_n$ onto E_0 forms a CM Type Φ_0 of E_0 . Thus, (Φ, E) comes from the extension of (Φ_0, E_0) , so Φ is not primitive. $\square$

We now give classification up to isomorphism. To simplify discussion, we give the E -action.

 Proposition 14 (Classification up to isomorphism).

Let E be a CM field of degree $2n$. Then there is a bijection between

$$\left\{ \begin{array}{c} \text{simple CM abelian variety } A \text{ of dimension } n \\ \text{isomorphism } i : E \rightarrow \text{End}^0 A \end{array} \right\} \simeq \frac{\left\{ \begin{array}{c} \text{primitive CM type } \Phi \text{ of } E \\ \text{lattice } L \subset E \end{array} \right\}}{L \sim \alpha L \text{ for all } L \text{ and } \alpha \in E}.$$

► *Proof.* The only new thing that we need to check is that given two abelian varieties A and B with the given actions $i : E \rightarrow \text{End}^0 A$ and $j : E \rightarrow \text{End}^0 B$, then $(A, i) \simeq (B, j)$ if and only if the corresponding CM types are the same and the corresponding lattices L and M are in the same class.

To see this, let $A = \mathbb{C}^n / \Lambda_A$ and $B = \mathbb{C}^n / \Lambda_B$. First, note that the action of E on $\mathbb{C}^n$ uniquely determines the CM type. Next, note that an isomorphism from $A \rightarrow B$ is a map from $\mathbb{C}^n \rightarrow \mathbb{C}^n$ that preserves the E -action on $\Lambda_A \otimes_{\mathbb{Z}} \mathbb{Q}$ and $\Lambda_B \otimes_{\mathbb{Z}} \mathbb{Q}$. Since the E -dimension of both spaces are 1, we get that this map must be multiplication by α for some α . Thus $\Lambda_B = \alpha \Lambda_A$, which implies that $M = \alpha L$. Conversely, the isomorphism is given by multiplication by α . $\square$

What about non-semisimple abelian varieties? We have the following.

 Proposition 15 (Classification of CM Abelian Varieties).

Let E be a CM algebra (i.e., product of CM fields) with dimension $2n$.

(a) There is a bijection between

$$\frac{\left\{ \begin{array}{c} \text{CM abelian variety } A \text{ of dimension } n \\ \text{inclusion } i : E \hookrightarrow \text{End}^0 A \end{array} \right\}}{\text{isogeny}} \simeq \{\text{CM type } \Phi \text{ of } E\}.$$

(b) There is a bijection between

$$\left\{ \begin{array}{l} \text{CM abelian variety } A \text{ of dimension } n \\ \text{inclusion } i : E \hookrightarrow \text{End}^0 A \end{array} \right\} \simeq \frac{\left\{ \begin{array}{l} \text{CM type } \Phi \text{ of } E \\ \text{lattice } L \subset E \end{array} \right\}}{L \sim \alpha L \text{ for all } L \text{ and } \alpha \in E}.$$

References

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