Fundamental Theorem of CM over a Reflex Field

Mark Saengrungkongka

November 1, 2026

First, we fix some notations that will be throughout this handout.

- Let A be a simple abelian variety with complex multiplication with CM field E and CM Type Φ.
- Suppose that A has a CM over K. In particular, K contains the reflex field E^* [Mil20, Prop. 7.11].
- We now **assume that End A contains the ring of integers $\mathcal{O}_E$.**
- For prime ℓ, we define the Tate modules

$$T_\ell A = \lim_{n\geq 0} A[\ell^n] \qquad V_\ell A = T_\ell A \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$$
$$\widehat{T}A = \prod_{\ell \text{ prime}} T_\ell A \qquad \widehat{V}A = \widehat{T}A \otimes_{\mathbb{Z}} \mathbb{Q}$$

  Recall also that $T_\ell A$, $V_\ell A$, $\widehat{T}A$, and $\widehat{V}A$ are free modules of rank 1 over $\mathcal{O}_E \otimes_{\mathbb{Z}} \mathbb{Z}_\ell$, $E \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$, $\widehat{\mathcal{O}_E}$, and $\mathbb{A}_E^\infty$, respectively [Mil20, Prop. 7.3].
- For each ideal $\mathfrak{a} \subset \mathcal{O}_E$, we let $\lambda^\mathfrak{a} : A \to A^\mathfrak{a}$ denote the $\mathfrak{a}$-multiplication.
- For any field k, let $\mathrm{Gal}_k = \mathrm{Gal}(\overline{k}/k)$. We let

$$\rho : \mathrm{Gal}_K \ \to\ \mathbb{A}_E^{\infty,\times} \subseteq \mathrm{GL}(\widehat{V}A)$$

  be the Galois representation. The image is contained in $\mathbb{A}_E^{\infty,\times}$ because every element in the image commutes with everything in E, so it must be in $\mathbb{A}_E^{\infty,\times}$ (cf. [Mil20, Cor. 7.4]). Since $\mathbb{A}_E^{\infty,\times}$ is commutative, ρ factors through $\mathrm{Gal}_K^{\mathrm{ab}}$.
- For any $\sigma \in \mathrm{Gal}_\mathbb{Q}$, we let ${}^\sigma A = A \otimes_{\overline{\mathbb{Q}}} \overline{\mathbb{Q}}_\sigma$, where we regard $\overline{\mathbb{Q}}_\sigma$ as $\overline{\mathbb{Q}}$-vector space by $a \cdot b = \sigma(a)b$ where $\cdot$ is the scalar multiplication in $\overline{\mathbb{Q}}$.
- For any number field k, let

$$\mathrm{Art}_k : \mathbb{A}_k^\times \to \mathrm{Gal}_k^{\mathrm{ab}}$$

  be the inverse of the reciprocity map in class field theory. We recall that
  - Art_k sends uniformizer to the **inverse** of the Frobenius element.
  - The kernel of Art_k is $\overline{k^\times k_\infty^{\times,0}}$.

The goal for these notes is to study the Galois action in CM abelian variety. We are primarily interested in two questions.

(a) For $\sigma \in \mathrm{Gal}_\mathbb{Q}$, we have the map $\sigma : A \to {}^\sigma A$, which induces a map between Tate modules $\sigma : \widehat{T}A \to \widehat{T}({}^\sigma A)$. Can we describe this map betwen Tate modules?

(b) For $\sigma \in \mathrm{Gal}_{\mathbb{Q}}$, the variety ${}^{\sigma}A$ has CM with type $(E, \sigma\Phi)$. Can we describe the relation between the ideal lattices that define ${}^{\sigma}A$ and A?

Let us explain why an answer to (a) implies an answer to (b). Suppse that $A \simeq \mathbb{C}^{\Phi}/\mathfrak{a}$. We fix an identification $\widehat{V}A \simeq \mathbb{A}_E^{\infty}$ and $\widehat{V}({}^{\sigma}A) \simeq \mathbb{A}_E^{\infty}$ induced by the E-bases of $H_1(A, \mathbb{Q})$ and $H_1({}^{\sigma}A, \mathbb{Q})$. Then this map is given by a multiplication by an element in $\alpha \in \mathbb{A}_E^{\infty}$. This can be transformed to a fractional ideal $\mathfrak{b}$ through the map

$$\begin{aligned} \mathbb{A}_E^{\infty,\times} &\to \mathcal{I}_E \\ (\ldots, \underset{\mathfrak{p}}{\alpha_{\mathfrak{p}}}, \ldots) &\mapsto \prod_{\mathfrak{p}} \mathfrak{p}^{v_{\mathfrak{p}}(\alpha_{\mathfrak{p}})}. \end{aligned}$$

Let α maps to $\mathfrak{b} \in \mathcal{I}_E$. Then we have that ${}^{\sigma}A \simeq \mathbb{C}^{\sigma\Phi}/\mathfrak{a}\mathfrak{b}$.

We note that although the identification is only unique up to $E^{\times}$, which changes $\mathfrak{b}$ by a principal ideal anyways.

For the first half of these notes, we focus on understanding question (a) for $\sigma \in \mathrm{Gal}_{E^*}$ (i.e., elements fixing E^*). This is a simpler case because ${}^{\sigma}A$ will have the same CM type as A.

§1 The Statement over Reflex Field

We want to extend our Galois representation map $\rho : \mathrm{Gal}_K \to \mathbb{A}_E^{\infty,\times}$ to the map $\eta : \mathrm{Gal}_{E^*} \to \mathbb{A}_E^{\infty,\times}/E^{\times}$.

Definition 1.

We define the map

$$\eta : \mathrm{Gal}_{E^*} \to \mathbb{A}_E^{\infty,\times}/E^{\times}$$

as follows: choose an E-linear isogeny $\alpha : A \to {}^{\sigma}A$. Then we let $\eta(\sigma)$ be such that the following diagram commutes

$$\widehat{V}A \xrightarrow{\alpha} \widehat{V}({}^{\sigma}A) \xrightarrow{\eta(\sigma)} \widehat{V}({}^{\sigma}A) \ , \qquad \widehat{V}A \xrightarrow{\sigma} \widehat{V}({}^{\sigma}A)$$

Notice that a choice of α can be modified up to multiplication by $E^{\times}$, so $\eta(\sigma)$ is only determined modulo $E^{\times}$, explaining the quotient by $E^{\times}$ in the target.

This map factors through $\mathrm{Gal}_{E^*}^{\mathrm{ab}}$, giving $\eta : \mathrm{Gal}_{E^*}^{\mathrm{ab}} \to \mathbb{A}_E^{\infty,\times}/E^{\times}$. We leave as an exercise to the reader to check that η is continuous and is a group homomorphism. The latter notably needs to fact that $\mathfrak{a}$-multiplication is Galois-equivariant.

Theorem 2 (The Main Theorem over Reflex Field)**.**

The following diagram commutes.

$$\begin{array}{ccc} \mathrm{Gal}_{E^*}^{\mathrm{ab}} & \xrightarrow{\eta} & \mathbb{A}_E^{\infty,\times}/E^{\times} \\ & \nwarrow^{\mathrm{Art}_{E^*}} & \uparrow{\scriptstyle \mathrm{N}_{\Phi}} \\ & & \mathbb{A}_{E^*}^{\infty,\times}/(E^*)^{\times} \end{array}$$

where N_{Φ} denotes the reflex norm.

In particular, this theorem determines Galois representation up to $E^\times$ by for any $\sigma \in \mathrm{Gal}_{E^*}^{\mathrm{ab}}$, we find an element $\alpha \in \mathbb{A}_{E^*}^{\infty,\times}/(E^*)^\times$ that correspond to σ under the Artin map. Then we take the reflex norm to obtain an element in $(\mathbb{A}_E^{\infty,\times})/E^\times$ that gives the entire representation.

§2 The Shimura-Taniyama Formula

The Shimura-Taniyama formula provides access to how the Frobenius element acts on the Tate module. We begin with the form of the formula stated in the previous talk.

Theorem 3 (Shimura-Taniyama Formula)**.**

Suppose that K contains all Galois conjugates of E. Let $\mathfrak{P}$ be a prime ideal over K and $(p) = \mathfrak{P} \cap \mathbb{Z}$. Suppose that p is unramified over K and A has a good reduction at $\mathfrak{P}$.

(a) There exists an element $\pi \in \mathcal{O}_E$ such that the action of π on the reduction of A modulo $\mathfrak{P}$ is the same as the Frobenius map.

(b) We have
$$(\pi) = (\mathrm{N}_\Phi(\mathfrak{P}))$$
as ideals in $\mathcal{O}_E$.

To prove this theorem, we need the following lemma about differentials.

Lemma 4.

Let k be an algebraically closed field with characteristic $p > 0$. Let $\alpha : A \to B$ be an isogeny of abelian varieties over k. Suppose that $\alpha^*(k(B)) \supset k(A)^{p^m}$ for some m. Let $d = \dim(\mathrm{Ker}(T_0 A \to T_0 B))$ (where $T_0 A$ and $T_0 B$ are tangent spaces). Then $\deg \alpha \le p^{md}$.

▶ *Proof (skipped during talk).* Let $L = k(A)$ and $K = k(B)$, so L is an extension of K. Let $x_1, \dots, x_n \in L$ be a minimal set of generators of L as K-algebra. Since $x_i^{p^m} \in K$, we get that $\deg \alpha = [L : K] \le p^{mn}$. Thus, it suffices to show that $n \le d$.

We note that $\dim \Omega^1_{L/K} = d$. For each i, each extension
$$L/\underbrace{K(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)}_{=K_i}$$
is nontrivial (since the generating set is minimal) and purely inseparable. Thus, there exists a K-linear derivation $D_i : L \to L$ such that $D_i(x_i) = 1$ but $D_i(x_j) = 0$ for all $j \ne i$ by sending everything in K_i to 0. Since $D_1, \dots, D_n$ are independent, we get that $d = \dim \Omega^1_{L/K} \ge n$. □

▶ *Proof of Theorem 3.* (a) This was done last time. Since A has a good reduction at $\mathfrak{P}$, the reduction map $\mathrm{End}\, A \to \mathrm{End}\, \widetilde{A}$ is injective, since the reduction $\widetilde{\phi} \in \mathrm{End}\, \widetilde{A}$ of $\phi \in \mathrm{End}\, A$ uniquely determines the action of ϕ in the torsion point $A[m]$ for all m coprime to $\mathfrak{p}$, which implies that it determines the whole ϕ (since torsion points is Zariski dense).

The Frobenius endomorphism commutes with everything in $\mathrm{End}\, A$, so it is in $\mathrm{End}^0\, A$. Being integral, it is in $\mathrm{End}\, A$.

(b) Let $\mathcal{A}$ be a smooth model of A over $\mathcal{O}_{\mathfrak{P}}$, and let $\widetilde{A}$ be its reduction to $k = \mathcal{O}_K/\mathfrak{P}$. Then we have that

$$\begin{aligned} T &:= T_0 A \simeq \bigoplus_{\varphi \in \Phi} K_\varphi \\ \mathcal{T} &:= T_0 \mathcal{A} \simeq \bigoplus_{\varphi \in \Phi} (\mathcal{O}_{\mathfrak{P}})_\varphi \\ \widetilde{T} &:= T_0 \widetilde{A} = \mathcal{T}/\mathfrak{P}\mathcal{T} = \bigoplus_{\varphi \in \Phi} k_\varphi, \end{aligned}$$

where the superscript φ indicates that E acts on K_φ by $a \cdot b = \varphi(a)b$ (and similarly to other modules). Since $\deg \pi = \mathrm{N}_{E/\mathbb{Q}}\pi$ is a power of p, we get that (π) is only divisible by primes above p. Thus, we get

$$(\pi) = \prod_{\mathfrak{p} \mid p} \mathfrak{p}^{m_\mathfrak{p}}$$

(as ideals of E). We let

$$\Phi_\mathfrak{p} = \{\varphi \in \Phi : \varphi^{-1}\mathfrak{P} = \mathfrak{p}\}.$$

Let h be the class number of E, and let $(\gamma_\mathfrak{p}) = \mathfrak{p}^{hm_\mathfrak{p}}$, so

$$(\pi^h) = \prod_{\mathfrak{p} \mid p} (\gamma_\mathfrak{p}).$$

We bound the degree of both sides. We note that

$$\begin{aligned} \mathrm{Ker}(d\widetilde{\gamma}_\mathfrak{p} : \widetilde{\mathcal{T}} \to \widetilde{\mathcal{T}}) &= \mathrm{Span}\{e_\varphi : \varphi(\gamma_v) \in \mathfrak{P}\} \\ &= \mathrm{Span}\{e_\varphi : \gamma_v \in \varphi^{-1}\mathfrak{P}\} \\ &= \mathrm{Span}\{e_\varphi : \mathfrak{p}^{hm_\mathfrak{p}} \subset \varphi^{-1}\mathfrak{P}\} \\ &\subseteq \mathrm{Span}\{e_\varphi : \varphi^{-1}\mathfrak{P} = \mathfrak{p}\} = \mathrm{Span}\{e_\varphi : \varphi \in \Phi_\mathfrak{p}\}. \end{aligned}$$

Therefore, $\dim \mathrm{Ker}(d\widetilde{\gamma}_\mathfrak{p}) \leq \#\Phi_\mathfrak{p}$. On the other hand, since $\gamma_\mathfrak{p}$ divides π^h, we get that $\gamma_p^*(k(\widetilde{A})) \supset k(\widetilde{A})^{p^{hd}}$, where $d = [k : \mathbb{F}_p]$. Thus, by the previous lemma, we have

$$\deg \gamma_\mathfrak{p} \leq p^{hd\,(\#\Phi_\mathfrak{p})}.$$

Thus, we have

$$p^{hdg} = \deg \pi^h = \prod_{\mathfrak{p} \mid p} \deg \gamma_\mathfrak{p} \leq p^{hd \sum_{\mathfrak{p} \mid p} (\#\Phi_\mathfrak{p})} = p^{hdg},$$

so everything is an equality! In particular, we deduce that $\mathrm{N}_{E/\mathbb{Q}}(\gamma_\mathfrak{p}) = \deg \gamma_\mathfrak{p} = p^{hd(\#\Phi_\mathfrak{p})}$, so by taking out a power of h, we have

$$\begin{aligned} \mathrm{N}_{E/\mathbb{Q}}\mathfrak{p}^{m_\mathfrak{p}} = p^{d(\#\Phi_\mathfrak{p})} = \prod_{\varphi \in \Phi_\mathfrak{p}} p^d &= \prod_{\varphi \in \Phi_\mathfrak{p}} \mathrm{N}_{K/\mathbb{Q}}\mathfrak{P} \\ &= \prod_{\varphi \in \Phi_\mathfrak{p}} \mathrm{N}_{E/\mathbb{Q}} \left(\varphi^{-1} \left(\mathrm{N}_{K/\varphi E}\mathfrak{P}\right)\right) \\ &= \mathrm{N}_{E/\mathbb{Q}} \left(\prod_{\varphi \in \Phi_\mathfrak{p}} \varphi^{-1} \left(\mathrm{N}_{K/\varphi E}\mathfrak{P}\right) \right) \end{aligned}$$

which implies that

$$\mathfrak{p}^{m_\mathfrak{p}} = \prod_{\varphi \in \Phi_\mathfrak{p}} \varphi^{-1} \left(\mathrm{N}_{K/\varphi E}\mathfrak{P}\right),$$

since both sides are powers of $\mathfrak{p}$, so it suffices to compare exponents. Finally, mulitplying this across all $\mathfrak{p}$ gives

$$(\pi) = \prod_{\mathfrak{p}\mid p} \mathfrak{p}^{m_{\mathfrak{p}}} = \prod_{\varphi\in\Phi} \varphi^{-1}\left(\mathrm{N}_{K/\varphi E}\mathfrak{P}\right) = \mathrm{N}_{\Phi,K}(\mathfrak{P}),$$

as desired. □

Theorem 3 provides access to only the Frobenius elements that fix K. We now extend this to the elements that fixes E^*.

Corollary 5.

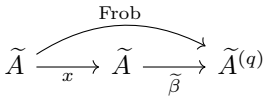
Suppose that K is Galois and contains all conjugates of E. Let $\mathfrak{P}$ be a prime ideal over K, $\mathfrak{p} = \mathfrak{P} \cap \mathcal{O}_{E^*}$, and $(p) = \mathfrak{p} \cap \mathbb{Z}$. Suppose that p is unramified over K and A has a good reduction at $\mathfrak{P}$.

(a) Let $\sigma = \mathrm{Frob}_{\mathfrak{P}/\mathfrak{p}} \in \mathrm{Gal}(K/E^*)$. Then there exists an $\mathfrak{a}$-multiplication $\alpha : A \to {}^{\sigma}A$ such that the reduction modulo $\mathfrak{P}$ is the q-power Frobenius map $\widetilde{\alpha} : \widetilde{A} \to \widetilde{A}^{(q)}$ where $q = (\mathcal{O}_{E^*} : \mathfrak{p})$.

(b) We have $\mathfrak{a} = \mathrm{N}_{\Phi}(\mathfrak{p})$.

▶ *Proof.* (a) Note that ${}^{\sigma}A$ and A have the same CM Type, so they are isogenous, which implies that there exists an $\mathfrak{b}$-multiplication $\beta : A \to {}^{\sigma}A$. Reducing modulo $\mathfrak{P}$ gives a $\mathfrak{b}$-multiplication $\widetilde{\beta} : \widetilde{A} \to \widetilde{A}^{(q)}$.

In the isogeny category, the Frobenius map $\widetilde{A} \to \widetilde{A}^{(q)}$ is a composition of $\widetilde{\beta}$ and an element $x \in \mathrm{End}\,\widetilde{A}$.



Since x commutes with every element in $\mathrm{End}\,A$, we get that $x \in \mathrm{End}^0 A$. Now, let $b_1, \ldots, b_n$ be generators of $\mathfrak{b}$. Then $b_i x \in \mathrm{End}\,\widetilde{A}$ for all A. Moreover, since $b_i x$ commutes with every element in $\mathrm{End}\,A$, we get that $b_i x \in \mathrm{End}\,A$ for all i. Thus, $x \in \mathfrak{b}^{-1}$. Finally, the composition of x and the $\mathfrak{b}$-multiplication gives a morphism $A \to {}^{\sigma}A$ whose reduction modulo $\mathfrak{p}$ is Frobenius.

(b) Let $f = [\kappa(\mathfrak{P}) : \kappa(\mathfrak{p})]$ be the degree of residue field. Note that

$$\pi = {}^{\sigma^{f-1}}\alpha \circ \cdots \circ {}^{\sigma}\alpha \circ \alpha,$$

and the right hand side is $\mathfrak{a}^f$-multipcation, so

$$\mathfrak{a}^f = (\pi) = \mathrm{N}_{\Phi,K}(\mathfrak{P}) = \mathrm{N}_{\Phi,E^*}(\mathrm{N}_{K/E^*}(\mathfrak{P})) = \mathrm{N}_{\Phi}(\mathfrak{p}^f) = \mathrm{N}_{\Phi}(\mathfrak{p})^f,$$

which implies that $\mathfrak{a} = \mathrm{N}_{\Phi}(\mathfrak{p})$. □

§3 Proof of the Main Theorem

Let us now head towards the proof of Theorem 2. We have to show that the diagram

$$\begin{array}{ccc} \mathrm{Gal}_{E^*}^{\mathrm{ab}} & \xrightarrow{\eta} & \mathbb{A}_E^{\infty,\times}/E^\times \\ & \nwarrow^{\mathrm{Art}_{E^*}} & \uparrow{\scriptstyle \mathrm{N}_\Phi} \\ & & \mathbb{A}_{E^*}^{\infty,\times}/(E^*)^\times \end{array}$$

commutes. Corollary 5 describes how the Frobenius elements interact with the Tate module of A. However, there are two impending issues before we can get the proof.

- First, the reduction modulo $\mathfrak{P}$ does not give a complete information of how Galois element acts. It only allows us to deduce information about ℓ-power torsion for primes $\ell \neq p$.
- Second, the target space $\mathbb{A}_E^{\infty,\times}/E^\times$ is not Hausdorff (in other words, $E^\times$ is not closed in $\mathbb{A}_E^{\infty,\times}$), so we cannot directly take limit.

Let us look into the first issue in more detail. Suppose that we plug in the idele

$$j_{\mathfrak{p}} = (\dots, 1, \underset{\mathfrak{p}}{\pi_{\mathfrak{p}}^{-1}}, 1, \dots)$$

into the bottom set. Then the top-left entry is $\sigma = \mathrm{Frob}_{\mathfrak{p}}$. Let $\alpha : A \to {}^\sigma A$ be an $\mathfrak{a}$-multiplication that reduces to Frobenius by Corollary 5. In particlar, $\mathfrak{a} = \mathrm{N}_\Phi(\mathfrak{p})$.

- Over all rational prime $\ell \neq p$, we can use injectivity of torsion to conclude that $\mathrm{Frob}_{\mathfrak{p}}$ acts the same way as α over ℓ-power torsion. Therefore, $\eta(\sigma)$ is equal to 1 in components prime to p. Thus, the diagram commutes in the components prime to p.
- On the component at prime $\mathfrak{q}$ above p, we no longer know how σ acts on the p-power torsion $A[p^\bullet] \to {}^\sigma A[p^\bullet]$ other than that it is bijective. Since α acts by multiplication by an element with the same $\mathfrak{q}$-adic valuation as $\mathfrak{a}$, we get that $\eta(\sigma)$ acts by an element with the same valuation as $\mathfrak{a}^{-1} = \mathrm{N}_\Phi(\mathfrak{p})^{-1}$.

$$\widehat{V}A \xrightarrow[\text{same valuation as } \mathfrak{a}]{\alpha} \widehat{V}({}^\sigma A) \xrightarrow[\text{same valuation as } \mathfrak{a}^{-1}]{\eta(\sigma)} \widehat{V}({}^\sigma A), \qquad \widehat{V}A \xrightarrow[\text{unit}]{\sigma} \widehat{V}({}^\sigma A)$$

Thus, we get that the diagram commutes up to $\prod_{v|p} \mathcal{O}_{E,v}^\times$.

Fix any $a \in \mathbb{A}_{E^*}^{\infty,\times}/(E^*)^\times$, and let $\sigma = \mathrm{Art}_{E^*}(a) \in \mathrm{Gal}_{E^*}$. We have to show that

$$t = D(a) := \frac{\mathrm{N}_\Phi(a)}{\eta(\sigma)} \text{ is in } E^\times.$$

To do this, we want to approximate a to be close to product of $j_{\mathfrak{p}}$. We do this as follows: take a open subgroup U, and let $V = D^{-1}(U)$. Take a finite set S of rational prime that includes

- all ramified primes over K, E, and E^*;
- all primes at which A has a bad reduction;
- U contains $\prod_{v \nmid S} \mathcal{O}_{E,v}^\times$; and
- V contains $\prod_{v \nmid S} \mathcal{O}_{E^*,v}^\times$.

(The product runs through all places coprime to all primes in S.)

By weak approximation, we divide out by an element in $(E^*)^\times$ such that a is close enough to be in V at all primes in S. By taking out the (finitely many) non-unit factors in a, we write $a = bc$ where $b \in \prod_{v|S} V_v \times \prod_{v\nmid S} \mathcal{O}_{E^*,v}^\times \subset V$ and c is in the subgroup generated by $\{j_\mathfrak{p} : \mathfrak{p} \nmid S\}$.

Thus, $t = D(a) = D(b)D(c) \in U$ because both $D(b)$ and $D(c)$ are in U. As V varies, we know that $\boxed{t \in \overline{E^\times}}$.

We now have to deal with the second issue of that we cannot take limits directly. In order to finish the proof, we need some more information about t. We let F be the largest totally real subfield of E (so E is a totally imaginary quadratic extension of F). We prove the following.

Proposition 6.

We have that $t\bar{t} \in F^\times$.

▶ *Proof.* On one hand, norm of a is controlled by class field theory In particular, we note that $\mathrm{N}_\Phi(a)\overline{\mathrm{N}_\Phi(a)} = \mathrm{N}_{E^*/\mathbb{Q}}(a)$. Furthermore, by functoriality of Artin map, we ave

$$\mathrm{Art}_\mathbb{Q}(\mathrm{N}_{E^*/\mathbb{Q}}(a)) = \mathrm{Art}_{E^*}(a) = \sigma|_{\mathbb{Q}^{\mathrm{ab}}} = \mathrm{Art}_\mathbb{Q}(\chi_{\mathrm{cyc}}(\sigma)),$$

where $\chi_{\mathrm{cyc}} : \mathrm{Gal}_\mathbb{Q} \to \widehat{\mathbb{Z}}$ is the cyclotomic character. The last equalty can be easily justified by noting that it is true for when σ is Frobenius element. Since $\ker \mathrm{Art}_\mathbb{Q} = \mathbb{Q} \cdot \mathbb{R}_{>0}$, we get that

$$\mathrm{N}_\Phi(a)\overline{\mathrm{N}_\Phi(a)} = \chi_{\mathrm{cyc}}(\sigma) \cdot u \quad \text{for some } u \in \mathbb{Q}_{>0}.$$

On the other hand, $\eta(\sigma)$ is controlled by the Riemann form. Let $\psi_A : \widehat{V}A \times \widehat{V}A \to \mathbb{A}_\mathbb{Q}^\infty(1)$ be the Riemann form extended to adelic Tate module. Then we have

$$\begin{aligned}
\psi_A(\chi_{\mathrm{cyc}}(\sigma)\, x,\ y) &= \chi_{\mathrm{cyc}}(\sigma)\, \psi_A(x,\ y) && (\widehat{\mathbb{Z}}\text{-linearity}) \\
&= {}^\sigma\psi_A(x,\ y) && (\text{Weil pairing takes torsion points to roots of unity}) \\
&= \psi_{{}^\sigma A}({}^\sigma x,\ {}^\sigma y) && (\text{Galois equivariance}) \\
&= \psi_{{}^\sigma A}(\eta(\sigma)\,\alpha(x),\ \eta(\sigma)\,\alpha(y)) && (\text{definition of } \eta) \\
&= \psi_{{}^\sigma A}(\overline{\eta(\sigma)}\eta(\sigma)\,\alpha(x),\ \alpha(y)). && (\text{properties of Riemann form, check!})
\end{aligned}$$

Note that $\psi_{{}^\sigma A}(\alpha(x),\ \alpha(y))$ is also a Riemann form of A. Therefore, by classification of Riemann form, we get that

$$\psi_{{}^\sigma A}(\alpha(x),\ \alpha(y)) = \psi_A(vx,\ y).$$

for some $v \in E^\times$. Therefore, we have that $\psi_A(\chi_{\mathrm{cyc}}(\sigma)x,\ y) = \psi_A(v\,\overline{\eta(\sigma)}\eta(\sigma)\,x,\ y)$ for all x, y. Nondegeneracy of Riemann form implies

$$\eta(\sigma)\overline{\eta(\sigma)} = \frac{\chi_{\mathrm{cyc}}(\sigma)}{u},$$

which implies after comparing with the first equation that $t\bar{t} = uv \in E^\times$. Since $t\bar{t}$ is fixed by conjugation, it follows that $t\bar{t} \in F^\times$. □

The following lemma completes the proof of Theorem 2. The following proof of this lemma was suggested by ChatGPT.

Lemma 7.

If $t \in \overline{E^\times}$ and $t\bar{t} \in F^\times$, then $t \in E^\times$.

▶ *Proof.* We make a series of reductions.

- By dividing out by an element $e \in E^\times$ sufficiently close to t to match all vlauations, we can WLOG assume that $t \in \overline{\mathcal{O}_E^\times}$.
- By Dirichlet unit theorem, $\mathcal{O}_E^\times$ and $\mathcal{O}_F^\times$ have equal rank, so $\mathcal{O}_F^\times$ is a finite index subgroup of $\mathcal{O}_E^\times$. If $\mathcal{O}_E^\times = \bigcup a_i \mathcal{O}_F^\times$ where $a_i \in E^\times$, then $\overline{\mathcal{O}_E^\times} = \bigcup a_i \overline{\mathcal{O}_F^\times}$.

  Thus, without loss of generality (divide out by some a_i), assume that $t \in \overline{\mathcal{O}_F^\times}$.

Thus, we have $t = \bar{t}$, so $t^2 \in F^\times$. From continuity of valuation and that $t^2 \in \overline{\mathcal{O}_E^\times}$, we get that $t^2 \in \mathcal{O}_F^\times$.

Next, note that $\mathcal{O}_F^\times \simeq \{\pm 1\} \times \mathbb{Z}^r$ by Dirichlet's unit theorem. By **Chevalley's unit theorem** (see [Con]), all finite index subgroup of $\mathcal{O}_F^\times$ is a congruence subgroup. This implies that the congruence topology and the profinite topology of $\mathcal{O}_F^\times$ are the same, so

$$\overline{\mathcal{O}_F^\times} \simeq \widehat{\mathcal{O}_F^\times} = \{\pm 1\} \times \widehat{\mathbb{Z}}^r.$$

Say t correspond to (ε, z) where $\varepsilon \in \{\pm 1\}$ and $z \in \widehat{\mathbb{Z}}^r$. Then $2z \in \mathbb{Z}^r$, which means $z \in \mathbb{Z}^r$. Hence, $t \in \mathcal{O}_F^\times$. □

§4 A Little Bit About General Field

We now explain how to determine how elements in $\mathrm{Gal}_{\mathbb{Q}}$ acts. In order to do this, we have to define some maps.

Definition 8.

For any $\sigma \in \mathrm{Gal}_{\mathbb{Q}}$, we define $F_\Phi(\sigma) \in \mathrm{Gal}_E^{\mathrm{ab}}$ as follows: for each embedding $\varphi : E \to \mathbb{C}$, pick a field automorphism $w_\varphi : \mathbb{C} \to \mathbb{C}$ such that

$$w_\varphi|_E = \varphi \quad \text{and} \quad w_{\iota\varphi} = \iota w_\varphi,$$

for all φ, where ι denote the complex conjugation. Then we let

$$F_\Phi(\sigma) = \prod_{\varphi \in \Phi} w_{\sigma\varphi}^{-1} \sigma w_\varphi|_{E^{\mathrm{ab}}} \in \mathrm{Gal}_E^{\mathrm{ab}},$$

noting that $F_\Phi(\sigma)$ clearly fixes E. It's easy to check that this does not depend on the choice of w.

Finally, we define $f_\Phi(\sigma) \in \mathbb{A}_E^{\infty,\times}/E^\times$ to be the element such that

$$\mathrm{Art}_E(f_\Phi(\sigma)) = F_\Phi(\sigma) \quad \text{and} \quad f_\Phi(\sigma)\overline{f_\Phi(\sigma)} = \chi_{\mathrm{cyc}}(\sigma)E^\times.$$

(Existence and uniqueness are nontrivial.)

Theorem 9 (The Main Theorem over any field)**.**

For any $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and any representative $f \in \mathbb{A}_E^{\infty,\times}$ of $f_\Phi(\sigma)$, there exists an E-linear isomorphism $H_1(A, \mathbb{Q}) \simeq H_1({}^\sigma A, \mathbb{Q})$, which induces an isomorphism $\alpha : \widehat{V}A \to \widehat{V}({}^\sigma A)$ such that the following diagram commutes

$$\widehat{V}A \xrightarrow[\alpha]{} \widehat{V}({}^{\sigma}A) \xrightarrow[f]{} \widehat{V}({}^{\sigma}A) \ . \qquad \overset{\sigma}{\frown}$$

References

[Con] Keith Conrad. *The Congruence Subgroup Problem for Units*. URL: https://kconrad.math.uconn.edu/blurbs/gradnumthy/chevalleyunit.pdf (↑ p. 8).

[Mil20] James S. Milne. *Complex Multiplication (v0.10)*. www.jmilne.org/math/CourseNotes/cm.html. 2020 (↑ p. 1).