Introduction to Khovanov Homology

Mark Saengrungkongka

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§1 Introduction

Determining whether two knots (or more generally, links) are isotopic (i.e., equivalent under a smooth deformation) is a central problem in knot theory. The most common strategy to prove that two links L_1 and L_2 are not isotopic is to find a link invariant that outputs different values on L_1 and L_2. One example of a link invariant is **Jones polynomial**, which takes a link and outputs a polynomial in $\mathbb{Z}[q, q^{-1}]$. The definition of Jones polynomial is quickly reviewed in Section 2.1.

In a breakthrough paper [Kho00], Mikhail Khovanov manages to strengthen Jones polynomial by constructing several chain complexes of vector spaces such that

- reading off their Euler characteristic gives coefficients of the Jones polynomial; and
- reading off their homology groups gives a link invariant called **Khovanov homology**.

Similar to how homology groups of a topological space are stronger invariants than the Euler characteristic, the Khovanov homology is a strictly stronger invariant than Jones polynomial.

The goal of this paper is to introduce the reader to this construction. The main reference for this paper is [Bar02].

The organization of this paper is as follows. In Section 2, we briefly review the construction of Jones polynomial, a precursor to Khovanov homology, and provide some motivation on how to generalize it. Section 3 introduce all algebraic infrastructures needed in the construction of Khovanov homology, including graded vector spaces, chain complexes, and homology groups. Section 4 explains how to construct Khovanov's chain complex. Section 5 explains how to prove the invariance of Khovanov homology. Finally, Section 6 surveys some further results.

§2 Jones Polynomial

§2.1 Jones Polynomial

The reference for this subsection is [Lic97, Chapter 3], although our definition of Kauffmann bracket is different from the standard convention up to a simple variable substitution.

For any **unoriented** link diagram L, the **Kauffmann bracket**, denoted as $\langle L\rangle$, is a polynomial in $\mathbb{Z}[q, q^{-1}]$ satisfying the following three axioms:

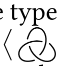
1. $\langle \emptyset \rangle = 1$, where $\emptyset$ is an empty diagram.
2. For any link L, we have $\langle L \sqcup \bigcirc \rangle = (q + q^{-1})\langle L\rangle$ (where $\sqcup$ denotes disjoint union and $\bigcirc$ is an unknot).

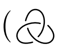
3. If $\times$, $\asymp$, and $\supset\subset$ are three link diagrams differ only at the shown crossing, then $\langle \times \rangle = \langle \asymp \rangle - q \langle \supset\subset \rangle$.

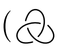
Kauffmann bracket is **not** a link invariant. It is invariant under Reidemeister moves R2 and R3, but not R1. However, a small modification, known as Jones polynomial, is a link invariant.

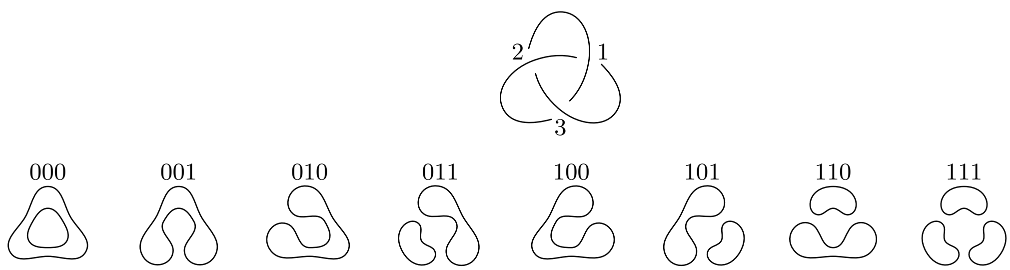
For any **oriented** link diagram L, if n_+ and n_- are the number of positive and negative crossings, then the **unnormalized Jones polynomial** $\hat{J}(L)$ and the **normalized Jones polynomial** (the usual one) $J(L)$ are polynomials in $\mathbb{Z}[q, q^{-1}]$ defined by

$$\hat{J}(L) = (-1)^{n_-} q^{n_+ - 2n_-} \langle L \rangle \quad \text{and} \quad J(L) = \hat{J}(L) \cdot (q + q^{-1})^{-1}. \tag{1}$$

Call $\asymp$ and $\supset\subset$ the **0-smoothing** and the **1-smoothing** of $\times$, respectively. Then, one can compute the Kauffmann bracket of a link with n crossings by applying (3) to reduce to computing the Kauffmann bracket of its 0-smoothing and 1-smoothing, both of which have $n-1$ crossings. Doing this for all crossings gives 2^n Kauffmann brackets, each have no crossing, and is determined by an element in $\{0,1\}^n$ specifying the type of smoothing at each crossing. An example for computing the Kauffmann bracket of a trefoil knot $\langle$  $\rangle$ is shown in Figure 1, and the result is $-q^6 + q^2 + 1 + q^{-2}$. In this knot, $n_+ = 3$ and $n_- = 0$, so we have

$\langle$  $\rangle = -q^6 + q^2 + 1 + q^{-2} \implies \hat{J}($  $) = -q^9 + q^5 + q^3 + q$

$\implies J($  $) = -q^8 + q^6 + q^2.$ (2)



000	001	010	011	100	101	110	111	sum
$(q+q^{-1})^2$	$-q(q+q^{-1})$	$-q(q+q^{-1})$	$q^2(q+q^{-1})^2$	$-q(q+q^{-1})$	$q^2(q+q^{-1})^2$	$q^2(q+q^{-1})^2$	$-q^3(q+q^{-1})^3$	
							$-q^6$	$-q^6$
			q^4		q^4	q^4	$-3q^4$	
q^2	$-q^2$	$-q^2$	$2q^2$	$-q^2$	$2q^2$	$2q^2$	$-3q^2$	q^2
2	-1	-1	1	-1	1	1	-1	1
q^{-2}								q^{-2}

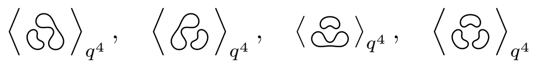
$\Big\langle$  $\Big\rangle = -q^6 + q^2 + 1 + q^{-2}$

Figure 1. Computing Kauffmann bracket of trefoil knot. Here, the 8 possible smoothings are shown in picture, and the contribution from each smoothing is shown right below the corresponding picture.

§2.2 Extending Jones Polynomial

In the computation of Kauffmann bracket, the resulting polynomial from 2^n smoothings were summed directly. The idea of Khovanov homology is to retain more information by instead of summing directly, we take more care in determining how should the result between different smoothing should cancel.

For example, in Figure 1, if one looks at the row consisting of coefficients of q^4, one will see that there are 4 smoothings that gives a nontrivial coefficient of q^4. One can strengthen this by not viewing those as an element of vector space of dimension 1 spanned by q^4 alone, but rather, viewing those as a vector space generated by 4 symbols[1]:



The problem is now that we need to carefully determine which symbols should we be cancelling. In particular, we must cancel just enough things so that the result is an invariant under Reidemeister moves. It turns out that the right construction for this is to make a *chain complex* out of everything related to q^4, and the *homology groups* will be a knot invariant called **Khovanov homology**. The terms chain complex and homology group will be introduced in Section 3.

Once we do this for coefficinet of q^n for every $n \in \mathbb{Z}$, we get infinitely many chain complexes, each indexed by $\mathbb{Z}$. A direct sum of these vector spaces / chain complexes can be viewed as a *graded vector space*, which is a tool to work with all coefficients at once. We will also introduce graded vector spaces in Section 3.

§3 Graded Vector Spaces and Chain Complexes

We now describe **chain complexes**, a powerful bookkeeping tool used in the definiton of Khovanov homology. To simplify discussion, we restrict ourselves to the case of vector spaces, but most of the discussion generalize to abelian groups and modules (and abelian categories).

§3.1 Graded Vector Space

Graded vector spaces allow us to work with all coefficients q^n for each $n \in \mathbb{Z}$ at once. Before we introduce graded vector spaces, we make the following conventions:

$$\boxed{\text{For this paper, all vector spaces are over } \mathbb{Q} \text{ and finite-dimensional.}}$$

We often write $\langle \bullet \rangle$ to denote the subspace generated by $\bullet$.

A **graded vector spaces** $V_\bullet$ is a direct sum $\bigoplus_{n\in\mathbb{Z}} V_n$ of vector spaces $\ldots, V_{-1}, V_0, V_1, \ldots$ indexed by $\mathbb{Z}$. The **i-th graded part** of $V_\bullet$ is V_i. An element in V_i is said to have **degree** i. Morphisms, kernels, images, subspaces, and quotients of graded vector spaces are defined by their (independent) behavior of the i-th graded part. For example, a morphism between graded vector spaces $\phi : V_\bullet \to W_\bullet$ is uniquely determined by morphisms of vector spaces $\phi_n : V_n \to W_n$ for each $n \in \mathbb{Z}$. One should think of graded vector spaces as a bag that holds many vector spaces at once.

Given a graded vector space $V_\bullet$, define the **q-dimension** $q\dim V_\bullet := \sum_{n\in\mathbb{Z}} q^n \dim V_n \in \mathbb{Z}[q, q^{-1}]$. We also define the **shift operator** $\bullet\{m\}$ that takes a graded vector space $V_\bullet$ to another graded vector space $V\{m\}_\bullet$ by $V\{m\}_i = V_{i+m}$.

§3.2 Tensor Products

Given two vector spaces V and W, the **tensor product** $V \otimes W$ is a new space spanned by symbols in the set $\{v \otimes w : v \in V, w \in W\}$ satisfying the following relations:

1. $v \otimes aw = av \otimes w = a(v \otimes w)$ for all $a \in \mathbb{Q}$, $v \in V$, and $w \in W$.
2. $(v_1 + v_2) \otimes w = (v_1 \otimes w) + (v_2 \otimes w)$ for all $v_1, v_2 \in V$ and $w \in W$.

[1]The actual Khovanov homology strengthen this even further by having the coefficient $-3q^4$ in the 111 smoothing come from 3 different basis elements. For the purpose of explaining motivation, we ignore this fact for now.

3. $v \otimes (w_1 + w_2) = (v \otimes w_1) + (v \otimes w_2)$ for all $v \in V$ and $w_1, w_2 \in W$.

If $(e_1, \ldots, e_m)$ and $(f_1, \ldots, f_n)$ are bases of V and W, respectively, then $V \otimes W$ has dimension mn and basis $\{e_i \otimes f_j\}$. We also define $V^{\otimes n} := \underbrace{V \otimes \cdots \otimes V}_{n\ V\text{'s}}$. We leave the interested reader to verify the following proposition:

Proposition 1.

For any three vector spaces U, V, W, we have $(U \otimes V) \otimes W = U \otimes (V \otimes W)$ and $(U \oplus V) \otimes W = (U \otimes W) \oplus (U \otimes W)$.

Given two graded vector spaces $V_\bullet$ and $W_\bullet$, the k-graded part of their **tensor product** $V_\bullet \otimes W_\bullet$ is

$$(V_\bullet \otimes W_\bullet)_k = \bigoplus_{i+j=k} V_i \otimes W_j. \tag{3}$$

A straightforward calculation shows that $q\dim(V_\bullet \otimes W_\bullet) = (q\dim V_\bullet)(q\dim W_\bullet)$.

Example 2.

We let $U_\bullet$ be the graded vector space with two basis elements: u_+ and u_- of degrees 1 and -1, respectively. (This notation will be used throughout the paper.) Hence, $q\dim U_\bullet = q + q^{-1}$. This space represents an unknot. Then, $U_\bullet^{\otimes 2}$ is a graded vector space with

- degree 2 part spanned by $u_+ \otimes u_+$.
- degree 0 part spanned by $u_+ \otimes u_-$ and $u_- \otimes u_+$.
- degree -2 part spanned by $u_- \otimes u_-$.

Thus, $q\dim U_\bullet^{\otimes 2} = q^2 + 2 + q^{-2} = (q + q^{-1})^2$. The space $U_\bullet^{\otimes 2}$ represents a disjoint union of two unknots. Analogously, the space $U_\bullet^{\otimes n}$ represents a disjoint union of n unknots. Notice how graded vector space is able to keep track of all coefficients at once.

§3.3 Chain Complexes

A **chain complex** (of graded vector spaces) $\mathcal{C}$ consists of

- a sequence of graded vector spaces $V_\bullet^{(i)}$ for $i \in \mathbb{Z}$; and
- for each $i \in \mathbb{Z}$, the **differential map** $\partial^{(i)} : V_\bullet^{(i)} \to V_\bullet^{(i+1)}$

satisfying $\partial^{(i)} \circ \partial^{(i-1)} = 0$ for all $i \in \mathbb{Z}$. The ***i*-th part** of $\mathcal{C}$ is $V_\bullet^{(i)}$. One should think of this as a bag that holds infinitely many chain complexes of vector spaces, each corresponding to the j-th graded part for each $j \in \mathbb{Z}$.

Let $\mathcal{C} = (V_\bullet^{(i)})_{i \in \mathbb{Z}}$ and $\mathcal{C}' = (W_\bullet^{(i)})_{i \in \mathbb{Z}}$ be two chain complexes. A **morphism between two chain complexes** $\phi : \mathcal{C} \to \mathcal{C}'$ consists of morphisms of graded vector spaces $\phi^{(i)} : V_\bullet^{(i)} \to W_\bullet^{(i)}$ such that the following diagram commutes.

$$\begin{tikzcd}
\mathcal{C}: & \cdots \arrow[r] & V_\bullet^{(0)} \arrow[r, "\partial^{(0)}"] \arrow[d, "\phi^{(0)}"] & V_\bullet^{(1)} \arrow[r, "\partial^{(1)}"] \arrow[d, "\phi^{(1)}"] & V_\bullet^{(2)} \arrow[r] \arrow[d, "\phi^{(2)}"] & \cdots \\
\mathcal{C}': & \cdots \arrow[r] & W_\bullet^{(0)} \arrow[r, "\partial^{(0)}"] & W_\bullet^{(1)} \arrow[r, "\partial^{(1)}"] & W_\bullet^{(2)} \arrow[r] & \cdots
\end{tikzcd}$$

If $V_\bullet^{(i)} \subseteq W_\bullet^{(i)}$ and $\phi^{(i)}$ is an inclusion map for all i, then we say that $\mathcal{C}$ is a **subchain complex** of $\mathcal{C}'$, denoted as $\mathcal{C} \subseteq \mathcal{C}'$. When this happens, we can define the **quotient chain complex** $\mathcal{C}'/\mathcal{C}$ whose i-th part is $W_\bullet^{(i)}/V_\bullet^{(i)}$. The differential maps in $\mathcal{C}'/\mathcal{C}$ are naturally inherited from those of $\mathcal{C}'$. (It is left to the interested reader to check that it is well-defined.)

For any chain complex $\mathcal{C} = (V_\bullet^{(i)})_{i\in\mathbb{Z}}$, we note that $\partial^{(i)} \circ \partial^{(i-1)} = 0$ is equivalent to $\operatorname{Im} \partial^{(i-1)} \subseteq \operatorname{Ker} \partial^{(i)}$. We thus define the ***i*-th homology** to be the quotient

$$H^i(\mathcal{C}) := \frac{\operatorname{Ker} \partial^{(i)}}{\operatorname{Im} \partial^{(i-1)}}. \tag{4}$$

There are two kinds of shift operators for a chain complex: the **degree shift** $\bullet\{m\}$ that shift the degree of each graded vector space by m (i.e., replace $V_\bullet^{(i)}$ by $V_\bullet^{(i)}\{m\}$), and the **height shift** $\bullet[m]$ that shifts the position of each vector space in the chain complex by m (i.e., replace $V_\bullet^{(i)}$ with $V_\bullet^{(i+m)}$).

The (graded) **Euler characteristic** of a chain complex is defined as

$$\chi(\mathcal{C}) := \sum_{n\in\mathbb{Z}} (-1)^n q\dim V_\bullet^{(n)} = \sum_{n\in\mathbb{Z}} (-1)^n q\dim H^n(\mathcal{C}). \tag{5}$$

(The second equality can be easily verified by simple applications of rank-nullity theorem.)

Remark 3.

We very briefly explain the relation to homology that is an invariant of a topological space. Let M be a surface, decomposed to vertices in set V, (directed) edges in set E, and faces in set F. Then, we define a chain complex

$$\mathcal{C}: \qquad \ldots \longrightarrow 0 \longrightarrow \mathbb{Q}F \xrightarrow{\partial_2} \mathbb{Q}E \xrightarrow{\partial_1} \mathbb{Q}V \longrightarrow 0 \longrightarrow \ldots,$$

consisting of free vector spaces generated by vertices, edges, and faces. The boundary map from faces is determined by summing all edges, paying attention to direction, and the boundary map from edges is determined by subtracting the two endpoints. Then, the homology groups $H_1(M, \mathbb{Q})$ and $H_2(M, \mathbb{Q})$ are homology group of $\mathcal{C}$ at $\mathbb{Q}E$ and $\mathbb{Q}F$. (In particular, they are $\operatorname{Ker} \partial_1 / \operatorname{Im} \partial_2$ and $\operatorname{Ker} \partial_2$, respectively.) The Euler characteristic of this complex is $V - E - F$, which coincide with the Euler characteristic of M.

To generalize this to arbitrary topological space, one need to replace $\mathcal{C}$ a more canonical choice of complex. In particular, the n-th vector space of the new complex will be the free vector space generated by uncountably many symbols, each corresponding to continuous map from n-simplex to M. See [Hat02, Ch. 2] or [Mil21, Ch. 1] for more discussion about homology.

§4 The Khovanov Cube

After a lengthy algebraic preliminary, we now model the computation of Jones polynomial into chain complex. Since the computation of Jones polynomial can be easily reduced to computing Kauffmann bracket, most part of this section is focused on developing a chain complex analogue of Kauffmann bracket. In particular, for a link diagram L, we will construct the chain complex $\mathcal{K}(L)$ such that

(i) the homology groups $H_\bullet(\mathcal{K}(L))$ is invariant under Reidemeister moves R2 and R3, and behaves well under the Reidemeister move R1;

(ii) the Euler characteristic of $\chi(\mathcal{K}(L))$ is equal to the Kauffmann bracket $\langle L\rangle$.

If we have $\mathcal{K}(L)$, then we perform degree shift by $n_+ - 2n_-$ and height shift by $-n_-$ to get

$$Kh(L) = \mathcal{K}(L)[-n_-]\{n_+ - 2n_-\} \tag{6}$$

so that the homology groups of $Kh(L)$ is a link invariant and the Euler characteristic of $Kh(L)$ is equal to the unnormalized Jones polynomial $\hat{J}(L)$. The overall construction is depicted in Figure 2, which the reader should refer to for the entire section.

Remark 4.

The original Khovanov homology uses chain complexes of abelian groups (instead of vector spaces), which gives a slightly stronger invariant due to possibility for torsion. For simplicity, we will work over $\mathbb{Q}$-vector spaces.

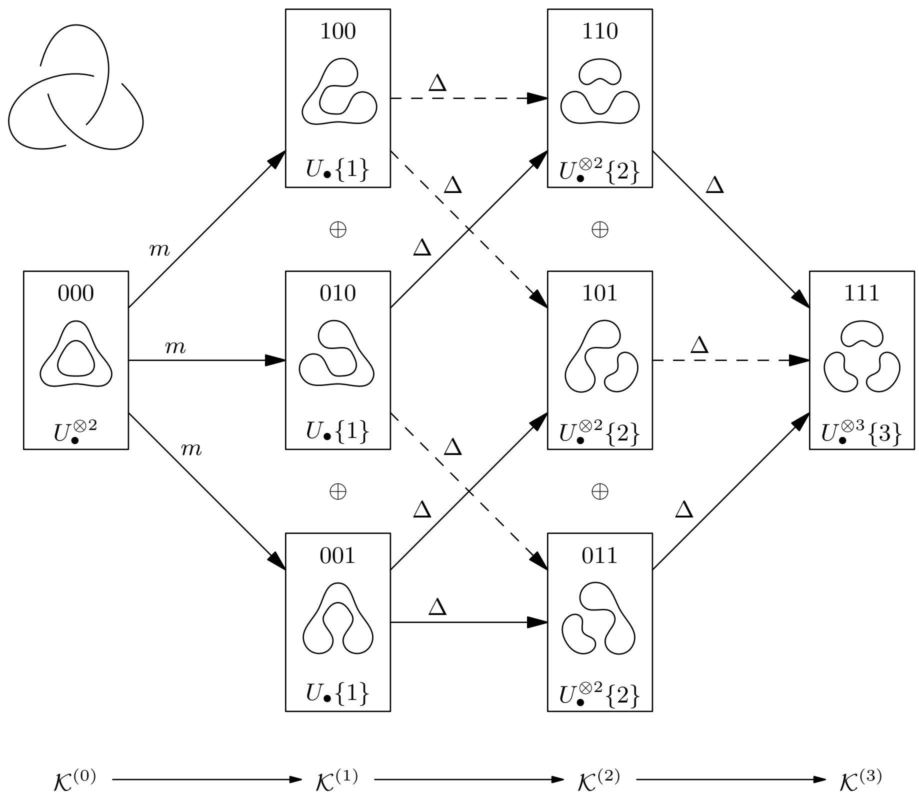


Figure 2. The Khovanov complex for trefoil knot. The arrows represent nontrivial maps between two blocks, labeled Δ and m according to whether the map splits or merges components, respectively. The dashed arrows are the ones assigned negative signs.

§4.1 Spaces

Let L be a link diagram with n crossings. Number the crossings $1, 2, \ldots, n$.

The calculation of Kauffmann bracket is build off of Kauffmann bracket of all possible 2^n smoothings. Similarly, the chain complex that we will build consists of 2^n **blocks**, each corresponding to a link obtained by smoothing all crossings of L. The r-th part of $\mathcal{K}(L)$ will be a direct sum of the $\binom{n}{r}$ blocks obtained by doing r 1-smoothings (and thus $n - r$ 0-smoothings).

To explain this in more detail, we set up the following notations.

- Let $\mathcal{X} = \{0,1\}^n$ be the hypercube. Thus, each element $x \in \mathcal{X}$ is an n-tuple of 0's and 1's.
- For each $x \in \mathcal{X}$, let $|x|$ denote the number of 1's in $x_1, \dots, x_n$.
- Let L_x be the link obtained by making smoothing of type x_i at the i-crossing for each $i \in \{1, \dots, n\}$.
- Let n_x be the number of components of L_x.
- Let $U_\bullet$ be the graded vector space with two basis elements: u_+ of degree 1 and u_- of degree -1 (same as Example 2).

Then, the r-th part of $\mathcal{K}(L)$ will be

$$\mathcal{K}(L)^{(r)} := \bigoplus_{\substack{x \in \mathcal{X} \\ |x| = r}} U_\bullet^{\otimes n_x}\{r\} \tag{7}$$

(i.e., each block is of the form $U_\bullet^{\otimes n_x}\{r\}$, and each $U_\bullet$ factor corresponds to a component of L_x). Observe that $q\dim U_\bullet^{\otimes n_x}\{r\} = q^r(q+q^{-1})^{n_x}$, which corresponds to contribution of L_x in the characteristic polynomial. The sign $(-1)^r$ that occurs when summing the Kauffmann brackets is accounted in the calculation of Euler characteristic.

✍ **Exercise 5.** Verify that the Euler characteristic of $\mathcal{K}(L)$ is equal to the Kauffmann bracket $\langle L \rangle$.

§4.2 Differential Maps: The Toy Cases

The construction of differential maps is more complicated. Thus, before going to the general construction, we illustrate it by giving several toy cases, which will serve as a building block for the general construction.

Example 6.

Suppose that $L = \bigcirc$ is an unknot. Then, the chain complex $\mathcal{K}(\bigcirc)$ looks like

$$\mathcal{K}(\bigcirc): \quad \dots \longrightarrow 0 \longrightarrow \underset{0 \text{ part}}{U_\bullet} \longrightarrow 0 \longrightarrow \dots$$

Thus, all differential maps are trivial. Since $n_+ = n_- = 0$, we have $Kh(\bigcirc) = \mathcal{K}(\bigcirc)$, so

$$H^i(Kh(\bigcirc)) = \begin{cases} U_\bullet & i = 0 \\ 0 & i \neq 0. \end{cases} \tag{8}$$

In other words, the degree 1 and -1 part of $H^0(Kh(\bigcirc))$ are $\mathbb{Q}$. All other homologies are 0.

Example 7.

Now suppose that $L = \bigcirc\!\!\bigcirc$ is a link diagram trivially isotopic to an unknot. Then, the 0-smoothing and 1-smoothing of L have 1 and 2 components, respectively. Thus, the chain complex, when broken down to various degrees, looks like

$$
\begin{array}{rcccccc}
 & & & \text{0-th part} & & \text{1-st part} & \\
\mathcal{K}(\bigcirc\!\!\bigcirc): & \ldots \longrightarrow 0 \longrightarrow & & U_\bullet & \longrightarrow & U_\bullet^{\otimes 2}\{1\} & \longrightarrow 0 \longrightarrow \ldots \\
\deg -1: & \ldots \longrightarrow 0 \longrightarrow & & \langle u_- \rangle & \longrightarrow & \langle u_- \otimes u_- \rangle & \longrightarrow 0 \longrightarrow \ldots \\
\deg 1: & \ldots \longrightarrow 0 \longrightarrow & & \langle u_+ \rangle & \longrightarrow & \langle u_+ \otimes u_-,\ u_- \otimes u_+ \rangle & \longrightarrow 0 \longrightarrow \ldots \\
\deg 3: & \ldots \longrightarrow 0 \longrightarrow & & 0 & \longrightarrow & \langle u_+ \otimes u_+ \rangle & \longrightarrow 0 \longrightarrow \ldots
\end{array}
\tag{9}
$$

From the previous example, we expect a nontrivial homology of $Kh_\bullet(\bigcirc\!\!\bigcirc)$ at degrees -1 and 1. Here, $n_+ = 0$ and $n_- = 1$, so the degree shift is by -2, and so we expect a nontrivial homology of $\mathcal{K}(\bigcirc\!\!\bigcirc)$ at degrees 1 and 3. In particular, the map at degree -1 should be an isomorphism. A natural choice for the differential map is to have

$$
u_- \mapsto u_- \otimes u_- \quad \text{and} \quad u_+ \mapsto (u_+ \otimes u_-) + (u_- \otimes u_+). \tag{10}
$$

One can check that this gives the equal Khovanov homology as above.

Example 8.

Now suppose that $L = \bigcirc\!\!\bigcirc$ is another link diagram trivially isotopic to an unknot. This time, the 0-smoothing and 1-smoothing of L have 2 and 1 components, respectively. Thus, the chain complex, when broken down to various degrees, looks like

$$
\begin{array}{rccccc}
 & & \text{0-th part} & & \text{1-st part} & \\
\mathcal{K}(\bigcirc\!\!\bigcirc): & \ldots \longrightarrow 0 \longrightarrow & U_\bullet^{\otimes 2} & \longrightarrow & U_\bullet\{1\} & \longrightarrow 0 \longrightarrow \ldots \\
\deg -2: & \ldots \longrightarrow 0 \longrightarrow & \langle u_- \otimes u_- \rangle & \longrightarrow & 0 & \longrightarrow 0 \longrightarrow \ldots \\
\deg 0: & \ldots \longrightarrow 0 \longrightarrow & \langle u_+ \otimes u_-,\ u_- \otimes u_+ \rangle & \longrightarrow & \langle u_- \rangle & \longrightarrow 0 \longrightarrow \ldots \\
\deg 2: & \ldots \longrightarrow 0 \longrightarrow & \langle u_+ \otimes u_+ \rangle & \longrightarrow & \langle u_+ \rangle & \longrightarrow 0 \longrightarrow \ldots
\end{array}
\tag{11}
$$

We expect a nontrivial homology of $Kh(\bigcirc\!\!\bigcirc)$ at degrees -1 and 1. Here, $n_+ = 1$ and $n_- = 0$, so the degree shift is by 1, and so we expect a nontrivial homology of $\mathcal{K}(\bigcirc\!\!\bigcirc)$ at degrees -2 and 0. In particular, the map at degree 2 should be an isomorphism. A natural choice for the boundary map is to have

$$
\begin{aligned}
u_+ \otimes u_- &\mapsto u_-, \qquad & u_- \otimes u_- &\mapsto 0, \\
u_- \otimes u_+ &\mapsto u_-, \qquad & u_+ \otimes u_+ &\mapsto u_+.
\end{aligned}
\tag{12}
$$

One can check that this gives the equal Khovanov homology as above.

§4.3 Differential Maps: The Construction

We now explain the construction of differential maps in general. Refer to Figure 2 for an illustration in the case of trefoil knot.

1. **Map along the edge of the cube.** In order to define the map $\partial^{(r)} : \mathcal{K}(L)^{(r)} \to \mathcal{K}(L)^{(r+1)}$, it suffices to define maps from $\partial_{xy} : U_\bullet^{\otimes n_x}\{r\} \to U_\bullet^{\otimes n_y}\{r+1\}$ for all $x, y \in \mathcal{X}$ such that $|x| = r$ and $|y| = r + 1$. Then, we can take the direct sum of those maps.

   If x and y do not form the edge of the hypercube (i.e., they differ in more than one position), then we declare that ∂_{xy} is zero map. Otherwise, links L_x and L_y differ by a change from 0-smoothing to 1-smoothing, or vice versa. Hence, it must be one of the following two possibilities.

   (a) **If L_y has one more component than L_x** (i.e., the change splits one component into two, labeled Δ in Figure 2.), then the construction is essentially Example Example 7. Recall that each factor in $U_\bullet^{\otimes n_y}\{r\}$ and $U_\bullet^{\otimes n_y}\{r+1\}$ correspond to a component of L_x and L_y. If a component remain intact after changing x to y, then we do not need to do anything (in fact, the basis of that component will have degree $r-1$ and $r+1$ in x, but r and $r+2$ in y, which do not overlap.)

Thus, we only need to define the map from the $U_\bullet$ factor corresponding to the component in L_x that got split into two $U_\bullet \otimes U_\bullet$ factors corresponding to two components in L_y. Drawing inspiration from (10), we define the map

$$\begin{aligned} & \partial_{xy} : U_\bullet\{r\} \to (U_\bullet \otimes U_\bullet)\{r+1\} \\ \text{by} \qquad & u_- \mapsto u_- \otimes u_- \\ & u_+ \mapsto (u_+ \otimes u_-) + (u_- \otimes u_+). \end{aligned} \tag{13}$$

(b) **If L_y has one less component than L_x** (i.e., the change merges two components into one, labeled m in Figure 2.), then the construction is essentially Example 8. Similar to (a), we only need to define a map to the two $U_\bullet \otimes U_\bullet$ factors corresponding to two components in y that got merged, resulting in a single factor $U_\bullet$ in x. Drawing upon (12), we define the map

$$\begin{array}{lllll} & \partial_{xy} : U_\bullet\{r\} & \to & (U_\bullet \otimes U_\bullet)\{r+1\} & \\ \text{by} & u_+ \otimes u_- & \mapsto & u_-, & u_- \otimes u_- \mapsto 0, \\ & u_- \otimes u_+ & \mapsto & u_-, & u_+ \otimes u_+ \mapsto u_+. \end{array} \tag{14}$$

2. **Sign adjustments.** It turns out that for every face in the cube, the four maps surrounding this face forms a commuting square. Unfortunately, this means that the composition of two consecutive boundary maps $\partial^{(r+1)} \circ \partial^{(r)}$ will not be zero map, violating definition of chain complex.

   To fix this, we need to make all squares *anticommute*. In particular, we will flip the sign of some of the maps ∂_{xy} to $-\partial_{xy}$. We choose to assign signs to each edge of $\mathcal{X}$ so that each face has exactly 1 or 3 negative signs, which will make the two paths along each square cancel each other, making the composition a zero map. Any assignment satisfying this condition will do the job. One such assignment is to let

$$\epsilon_{xy} = (-1)^{\sum_{i<j} x_i}, \quad \text{where } j \text{ is the unique index such that } x_j \neq y_j. \tag{15}$$

   and replace ∂_{xy} with $\epsilon_{xy}\partial_{xy}$.

Once $\mathcal{K}(L)$ is constructed, the complex $Kh(L)$ is constructed according to (6). The major result of Khovanov homology is the following.

Theorem 9.

The homology groups $H^i(Kh(L))$ is invariant when the three Reidemeister moves are applied on L. Hence, $H^i(Kh(L))$ is a link invariant.

§4.4 Example: Hopf Link

We demonstrate how to compute the Khovanov homology of a Hopf link. The four smoothings of a Hopf link is shown in Figure 3. When broken down to different degrees, the chain complex is

$$\begin{array}{rccccccc} & & \text{0-th part} & & \text{1-st part} & & \text{2-nd part} & \\ \mathcal{K}(\bigcirc\!\!\!\bigcirc) : & \longrightarrow & U_\bullet^{\otimes 2} & \longrightarrow & U_\bullet\{1\} \oplus U_\bullet\{1\} & \longrightarrow & U_\bullet^{\otimes 2}\{2\} & \longrightarrow \dots \\ \deg -2 : & \dots \longrightarrow & \langle u_- \otimes u_- \rangle & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow \dots \\ \deg 0 : & \dots \longrightarrow & \langle u_+ \otimes u_-,\ u_- \otimes u_+ \rangle & \longrightarrow & \langle u_- \rangle \oplus \langle u_- \rangle & \longrightarrow & \langle u_- \otimes u_- \rangle & \longrightarrow \dots \\ \deg 2 : & \dots \longrightarrow & \langle u_+ \otimes u_+ \rangle & \longrightarrow & \langle u_+ \rangle \oplus \langle u_+ \rangle & \longrightarrow & \langle u_+ \otimes u_-,\ u_- \otimes u_+ \rangle & \longrightarrow \dots \\ \deg 4 : & \dots \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \langle u_+ \otimes u_+ \rangle & \longrightarrow \dots \end{array} \tag{16}$$

Computing Khovanov homology at degree -2 and 4 are easy because all maps are zero maps. We now explain how to compute homology at degree 0.

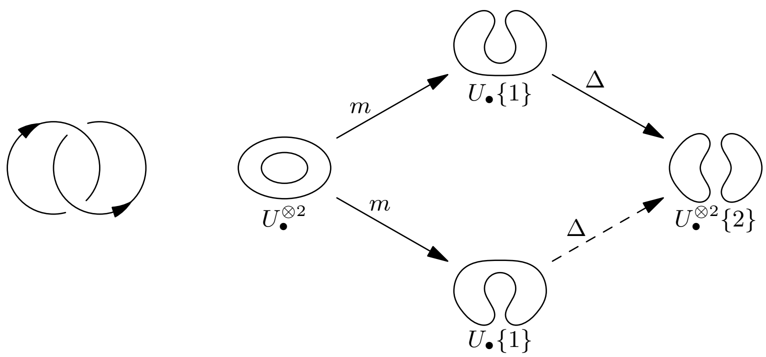


Figure 3. Hopf Link L (left), and the corresponding Khovanov chain complex $\mathcal{K}(L)$ (right).

- The zeroth homology H^0 is simply the kernel from the map from 0-th part to 1-st part, which is generated by $(u_+ \otimes u_-) - (u_- \otimes u_+)$, so the zeroth homology is $\mathbb{Q}$.
- The image of the map from 0-th to 1-st part is (u_-, u_-), while the kernel of the map from 1-st to 2-nd part is also (u_-, u_-), so the first homology H^1 is 0.
- The map from 1-st to 2-nd part is surjective, so the second homology is H^2 is 0.

One can perform a similar analysis to degree 2, and we get that the only nontrivial homology is H^2, which is $\mathbb{Q}$. Thus, the result can be summarized by the left of Table 1. If one orient the Hopf link as in Figure 3, we get $n_+ = 0$ and $n_- = 2$, so the Khovanov homology can be summarized in the right table.

	$H^\bullet(\mathcal{K}(L))$		
	H^0	H^1	H^2
deg -2	$\mathbb{Q}$		
deg 0	$\mathbb{Q}$		
deg 2			$\mathbb{Q}$
deg 4			$\mathbb{Q}$

	$H^\bullet(Kh(L))$		
	H^{-2}	H^{-1}	H^0
deg -6	$\mathbb{Q}$		
deg -4	$\mathbb{Q}$		
deg -2			$\mathbb{Q}$
deg 0			$\mathbb{Q}$

Table 1. Homology $H^\bullet(\mathcal{K}(L))$ (left) and Khovanov homology $H^\bullet(Kh(L))$ (right), where L is the Hopf link as in Figure 3. Missing entries are 0.

§5 Invariance under Reidemeister Moves

In order to prove Theorem 9, it suffices to show that $H^i(Kh(L))$ is invariant under the following three **Reidemeister moves**: (R1) turning to , (R2) turning to , and (R3) turning to . Here, we only allow the "right twist" variant of R1. The left twist R1 can be obtained by a sequence of R2 moves and right twist R1 moves.

§5.1 A Useful Algebraic Lemma

We use the following algebraic lemma that will allow us to manipulate chain complex. A chain complex is **acyclic** if all homology groups vanish.

Lemma 10.

Let $\mathcal{C}'$ be a subchain complex of $\mathcal{C}$.

(a) If $\mathcal{C}'$ is acyclic, then $H^i(\mathcal{C}/\mathcal{C}') = H^i(\mathcal{C})$ for all i.

(b) If $\mathcal{C}/\mathcal{C}'$ is acyclic, then $H^i(\mathcal{C}') = H^i(\mathcal{C})$ for all i.

▶ *Proof.* A slick proof for both parts is to apply the homology long exact sequence [Hat02, Theorem 2.16] induced from exact sequence $0 \to \mathcal{C}' \to \mathcal{C} \to \mathcal{C}/\mathcal{C}' \to 0$. Here we sketch a tedious elementary proof for (a). (Part (b) can be proven with a similarly tedious argument, which we omit.)

Let $\mathcal{C} = (V_\bullet^{(i)})_{i\in\mathbb{Z}}$ and $\mathcal{C}' = (W_\bullet^{(i)})_{i\in\mathbb{Z}}$ for graded vector spaces $V_\bullet^{(i)}$ and $W_\bullet^{(i)}$. The map $H^i(\mathcal{C}) \to H^i(\mathcal{C}/\mathcal{C}')$ arises naturally from quotient projection. We first show that it is injective. Suppose $[\alpha] \in V_\bullet^{(i)}/W_\bullet^{(i)}$ (where $\alpha \in \operatorname{Ker}(V_\bullet^{(i)} \to W_\bullet^{(i)})$) is equal to zero in the homology group $H^i(\mathcal{C}'/\mathcal{C})$, so $[\alpha] = [\partial\beta]$ for some $\beta \in V_\bullet^{(i-1)}$, which means that $\alpha - \partial\beta \in W_\bullet^{(i)}$. Since $\partial(\alpha - \partial\beta) = 0$ and $W_\bullet^{(i)}$ is acyclic, there exists $\gamma \in W_\bullet^{(i-1)}$ such that $\alpha - \partial\beta = \partial\gamma$. Hence, $\alpha \in \operatorname{Im}(V_\bullet^{(i-1)} \to V_\bullet^{(i)})$, so it has zero homology class in $H^i(\mathcal{C})$.

Next, we show that it is surjective. Suppose that $[\alpha] \in \operatorname{Ker}(V_\bullet^{(i)}/W_\bullet^{(i)} \to V_\bullet^{(i+1)}/W_\bullet^{(i+1)})$. Then, $[\partial\alpha] = 0$, so $\partial\alpha \in W_\bullet^{(i)}$. Since $\partial(\partial\alpha) = 0$ and $W_\bullet^{(i)}$ is acyclic, there exists $\beta \in W_\bullet^{(i)}$ such that $\partial\alpha = \partial\beta$. Then, $\partial(\alpha - \beta) = 0$, and $[\alpha - \beta] = [\alpha]$. Hence, $[\alpha] \in H^i(\mathcal{C}'/\mathcal{C})$ arises from $\alpha - \beta \in H^i(\mathcal{C})$. ☐

§5.2 Invariance Under "Right twist" R1.

We prove that Khovanov homology is invariant under the right twist R1 move: the move that turns to (with this specific orientation).

▶ *Proof of Invariance Under "Right Twist" R1.* Let $L' =$ and $L =$ be two link differ only at the shown portion. Suppose that L has n crossings. Then, the complex $\mathcal{K} = \mathcal{K}(\ )$ consists of:

- 2^n blocks corresponding to smoothings of ; and
- 2^n blocks corresponding to smoothings of .

Denote $\mathcal{K}(\ )$ by the complex $W_\bullet^{(0)} \to W_\bullet^{(1)} \to \cdots \to W_\bullet^{(n)}$. Each of the $W_\bullet^{(i)}$ is a direct sum of $\binom{n}{i}$ spaces corresponding to each smoothing. For each direct summand, we pull out the $U_\bullet$ tensor factor corresponding to the component where the R1 move happens. Thus, by Proposition 1, each $W_\bullet^{(i)}$ can be written as $V_\bullet^{(i)} \otimes U_\bullet$, where $V_\bullet^{(i)}$ is new space.

Now, since and differ by one component, the i-th part of the complex $\mathcal{K}(\ )$ is $V_\bullet^{(i)} \otimes U_\bullet \otimes U_\bullet$. Once we assemble $\mathcal{K}(\ )$ and $\mathcal{K}(\ )$ into a single complex $\mathcal{K}(\ )$, the part corresponding to $\mathcal{K}(\ )$ will be shifted by 1. Hence, we have the complex

$$\mathcal{K} = \mathcal{K}(\ ) : \begin{array}{ccccccc} \overset{\text{0-th part}}{V_\bullet^{(0)} \otimes U_\bullet \otimes U_\bullet} & \longrightarrow & \overset{\text{1-st part}}{V_\bullet^{(1)} \otimes U_\bullet \otimes U_\bullet} & \longrightarrow & \overset{\text{2-nd part}}{V_\bullet^{(2)} \otimes U_\bullet \otimes U_\bullet} & \longrightarrow & \cdots \\ & \searrow & \oplus & \searrow & \oplus & \searrow & \\ & & V_\bullet^{(0)} \otimes U_\bullet & \longrightarrow & V_\bullet^{(1)} \otimes U_\bullet & \longrightarrow & \cdots \end{array} \tag{17}$$

The horizontal arrows are inherited from maps $\partial^{(i)}$ directly, while the diagonal arrows maps as in (13) (with sign alternates). More specifically, the diagonal arrows are defined by

$$\begin{aligned} v \otimes u_+ \otimes u_- &\mapsto (-1)^i (v \otimes u_-), & v \otimes u_- \otimes u_- &\mapsto 0, \\ v \otimes u_- \otimes u_+ &\mapsto (-1)^i (v \otimes u_-), & v \otimes u_+ \otimes u_+ &\mapsto (-1)^i (v \otimes u_+). \end{aligned} \tag{18}$$

We will now apply Lemma 10. To do that, we define a subchain complex according to the following diagram (here $\langle u_+ \rangle$ means its span)

$$\mathcal{K}' : \quad \begin{array}{ccccccc} \overset{\text{0-th part}}{V_\bullet^{(0)} \otimes U_\bullet \otimes \langle u_+ \rangle} & \longrightarrow & \overset{\text{1-st part}}{V_\bullet^{(1)} \otimes U_\bullet \otimes \langle u_+ \rangle} & \longrightarrow & \overset{\text{2-nd part}}{V_\bullet^{(2)} \otimes U_\bullet \otimes \langle u_+ \rangle} & \longrightarrow & \dots \\ & \overset{\sim}{\searrow} & \oplus & \overset{\sim}{\searrow} & \oplus & \overset{\sim}{\searrow} & \\ & & V_\bullet^{(0)} \otimes U_\bullet & \longrightarrow & V_\bullet^{(1)} \otimes U_\bullet & \longrightarrow & \dots \end{array} \tag{19}$$

Notice that the diagonal arrow is an isomorphism because it simply removes the $\otimes u_+$ factor and multiply by $(-1)^i$. We now show that $\mathcal{K}'$ is acyclic. To see this, it suffices to show that kernel is contained in the image. Suppose that $x \otimes u_+ \in V_\bullet^{(i)} \otimes U_\bullet \otimes \langle u_+ \rangle$ and $y \in V_\bullet^{(i-1)} \otimes U_\bullet$ together maps to 0 (through the differential map). Then, $\partial x = 0$ and $(-1)^i x + \partial y = 0$. Thus, $x = (-1)^{i+1} \partial y$, and so (x, y) is in the image of $((-1)^{i+1} y, 0)$.

Thus, by Lemma 10 (a), we have that $H^i(\mathcal{K}) = H^i(\mathcal{K}/\mathcal{K}')$. However, the chain complex complex $\mathcal{K}/\mathcal{K}'$ is

$$\mathcal{K}/\mathcal{K}' : \quad \begin{array}{ccccccc} \overset{\text{0-th part}}{V_\bullet^{(0)} \otimes U_\bullet \otimes \langle u_- \rangle} & \longrightarrow & \overset{\text{1-st part}}{V_\bullet^{(1)} \otimes U_\bullet \otimes \langle u_- \rangle} & \longrightarrow & \overset{\text{2-nd part}}{V_\bullet^{(2)} \otimes U_\bullet \otimes \langle u_- \rangle} & \longrightarrow & \dots \\ & \searrow & \oplus & \searrow & \oplus & \searrow & \\ & & 0 & \longrightarrow & 0 & \longrightarrow & \dots \end{array} .$$

The factor u_- shifts the degree by -1. Therefore, we have that $H^i(\mathcal{K}(\;)) = H^i(\mathcal{K}'/\mathcal{K}) = H^i(\mathcal{K}(\;))\{-1\}$. Accounting for that n_+ increases by 1 after R1 move, we conclude that $H^i(Kh(\;)) = H^i(Kh(\;))$. $\square$

§5.3 Invariance under R2

The R2 move is the move that turns to . We prove that Khovanov homology remains invariant under this move.

▶ *Proof of Invariance under R2.* Let $L' =$ and $L =$ be two links differ at the shown position. If L has n crossings, then $\mathcal{K} = \mathcal{K}(\;)$ contains 2^{n+2} blocks, corresponding to the smoothings of , , , and (2^n each).

Denote $\mathcal{K}(\;)$ by the complex $W_\bullet^{(0)} \to W_\bullet^{(1)} \to \cdots \to W_\bullet^{(n)}$. Consider $\mathcal{K}(\;)$. By pulling out the three $U_\bullet$ factors corresponding to the three components shown in the diagram, we can assume that each vector space can be written in form $V_\bullet^{(i)} \otimes U_\bullet \otimes U_\bullet \otimes U_\bullet$. Accounting for the height shift, each space of the complex $\mathcal{K}(\;)$ and $\mathcal{K}(\;)$ is in form $V_\bullet^{(i)} \otimes U_\bullet \otimes U_\bullet$. Thus, $\mathcal{K}$ can be represented in diagram.

$$\begin{tikzcd} & V_\bullet^{(i)} \otimes {\color{red}U_\bullet} \otimes U_\bullet \otimes {\color{blue}U_\bullet} \arrow[dr, "m"] & \\ V_\bullet^{(i)} \otimes {\color{red}U_\bullet} \otimes U_\bullet \arrow[ur, "\Delta"] \arrow[dr] \arrow[rr, phantom, "\mathcal{K}"] & & V_\bullet^{(i)} \otimes U_\bullet \otimes {\color{blue}U_\bullet} \\ & W_\bullet^{(i)} \arrow[ur] & \end{tikzcd} \tag{20}$$

To simplify the diagram, we drew only one space for each of the smaller complexes. (Thus, there are actually four rows of spaces.) The horizontal coordinate of each space represents the height of that space in the chain complex. The link diagram above each space represent which smoothing it comes from. In addition, in the differential maps, the red and blue spaces were fixed.

We now consider the subcomplex $\mathcal{K}' \subset \mathcal{K}$ and the corresponding quotient complex

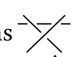
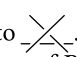
$$\begin{tikzcd}
& V_\bullet^{(i)} \otimes {\color{red}U_\bullet} \otimes \langle u_+ \rangle \otimes {\color{blue}U_\bullet} \arrow[dr, "\sim"] & & & V_\bullet^{(i)} \otimes {\color{red}U_\bullet} \otimes \langle u_- \rangle \otimes {\color{blue}U_\bullet} \arrow[dr] & \\
0 \arrow[ur, "\Delta"] \arrow[dr] \arrow[rr, phantom, "\substack{\mathcal{K}' \\ \text{acyclic}}"] & & V_\bullet^{(i)} \otimes U_\bullet \otimes {\color{blue}U_\bullet} & V_\bullet^{(i)} \otimes {\color{red}U_\bullet} \otimes U_\bullet \arrow[ur, "\Delta"] \arrow[dr] \arrow[rr, phantom, "\mathcal{K}/\mathcal{K}'"] & & 0 \;. \\
& 0 \arrow[ur] & & & W_\bullet^{(i)} \arrow[ur] &
\end{tikzcd} \tag{21}$$

By the same argument as the proof of R1, one can show that $\mathcal{K}'$ is acyclic. Thus, by Lemma 10 (a), $H^i(\mathcal{K}) = H^i(\mathcal{K}/\mathcal{K}')$. Now, we consider a subcomplex $\mathcal{K}''$ of $\mathcal{K}/\mathcal{K}'$.

$$\begin{tikzcd}
& 0 \arrow[dr] & & & V_\bullet^{(i)} \otimes {\color{red}U_\bullet} \otimes \langle u_- \rangle \otimes {\color{blue}U_\bullet} \arrow[dr] & \\
0 \arrow[ur] \arrow[dr] \arrow[rr, phantom, "\mathcal{K}''"] & & 0 & V_\bullet^{(i)} \otimes {\color{red}U_\bullet} \otimes U_\bullet \arrow[ur, "\sim"] \arrow[dr] \arrow[rr, phantom, "\substack{(\mathcal{K}/\mathcal{K}')/\mathcal{K}'' \\ \text{acyclic}}"] & & 0 \;. \\
& W_\bullet^{(i)} \arrow[ur] & & & 0 \arrow[ur] &
\end{tikzcd} \tag{22}$$

Again, one can show that $(\mathcal{K}/\mathcal{K}')/\mathcal{K}''$ is acyclic. Thus, by Lemma 10 (b), we get that $H^i(\mathcal{K}) = H^i(\mathcal{K}/\mathcal{K}') = H^i(\mathcal{K}'')$ for all i. However, $\mathcal{K}''$ is clearly isomorphic to $\mathcal{K}($$)$, so Khovanov homology remains the same under R2 move. ☐

§5.4 Invariance Under R3.

The move R3 is the move that turns  to . Unlike R1 and R2, both sides have a nontrivial crossing, and therefore one cannot prove invariance of R3 by simplifying one side to get the other. This makes the proof of R3 invariance more difficult than R1 and R2. We refer the reader to [Bar02, §3.5.5] for the proof.

§6 Modern Development

§6.1 How Strong is Khovanov Homology?

Since Khovanov's homology is designed to strengthen Jones Polynomial, we ask "do there exist two knots with the same Jones polynomial but different Khovanov homology?" (If not, then the invariant would be useless.) Fortunately, the answer is yes: [Bar02] computes Khovanov homology for all prime knots up to 10 crossings and discovered that knots 5_1 and 10_{132} are one such example.

Furthermore, Kronheimer and Mrowka [KM11] proved that Khovanov homology can detect an unknot, i.e., there is no knot with the same Khovanov homology as an unknot. In contrast, it is an open problem whether Jones polynomial can detect an unknot.

§6.2 An Abstract Point of View: Cobordism Category

In Bar-Natan's follow-up paper [Bar05], he abstracts the construction of Khovanov homology further by viewing the edges of the cube as a corbodism between two smoothings. This leads to **cobordism category**, whose objects are smoothings of a link and morphisms are cobordism between two smoothings. [Bar05] shows that if one modify this category by

- allowing for linear combination between cobordisms; and
- modding out by three specific relations,

then one can construct a chain complex in that category, whose homology groups are link invariant. To get a computable invariant, one can transform (i.e., take a functor) from that category to a category that we are more familiar with, such as modules and graded vector spaces. One such functor turns this category to graded vector spaces and results in Khovanov homology, as described above. However, there are more.

§6.3 Lee's Homology and Slice Invariant

In [Lee05], Lee introduces a new such functor, which leads to a chain complex $Kh'(L)$ that differ from Khovanov's chain complex in the three ways.

- The space corresponding to vertex x of the cube is instead $U^{\otimes n_x}$, where U is spanned by two symbols u_+ and u_-, and there is no grading.
- The map in (13) is instead

$$u_- \mapsto (u_- \otimes u_-) + (u_+ \otimes u_+), \quad u_+ \mapsto (u_+ \otimes u_-) + (u_- \otimes u_+). \tag{23}$$

- The map in (14) is instead

$$\begin{aligned} u_+ \otimes u_- &\mapsto u_-, & u_- \otimes u_- &\mapsto u_+, \\ u_- \otimes u_+ &\mapsto u_-, & u_+ \otimes u_+ &\mapsto u_+. \end{aligned} \tag{24}$$

(The difference is highlighted in blue.) [Lee05, Theorem 4.3] gives an explicit formula for the homology groups of Lee's complex $H^i(Kh'(L))$, and in particular, shows that $\sum_{i\in\mathbb{Z}} H^i(Kh'(L)) = 2^n$, where n is the number of components of L. If $L = K$ is a knot, then the only nontrivial homology is H^0, in which case $H^0(Kh'(K)) \simeq \mathbb{Q} \oplus \mathbb{Q}$.

What makes Lee's homology interesting, however, is the connection to Khovanov homology. By taking a obvious identification from Khovanov's homology (i.e., u_+ to u_+ and u_- to u_-), we get a spectral sequence[2] that goes from Khovanov's homology to Lee's homology. Rasmussen [Ras10, Proposition 3.3] proves that the two generators in Lee's homology $\mathbb{Q} \oplus \mathbb{Q}$ comes from elements of degree $s(K) \pm 1$ in Khovanov's homology, where $s(K) \in \mathbb{Z}$ is Rassmussen's ***s*-invariant**. If K is a slice knot, then $s(K) = 0$. Thus, s-invariant is a powerful technique to prove that a knot is not a slice, and played a key role in Piccirillo's solution [Pic20] to a longstanding open problem that Conway knot is not a slice knot.

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