The Necklace Splitting Problem

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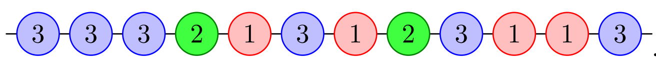
§1 Introduction

The goal of this article is to give a complete proof of the following result, due to Alon [Alo87].

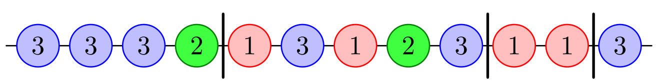
Theorem 1 (Necklace splitting).

There is a necklace consisting of beads arranged in a line. The beads come in n different colors, and for all $1 \le i \le n$, there are ka_i beads of the i-th color. Then it is possible to split this necklace using at most $n(k-1)$ cuts and reassemble them into k pieces so that for all $1 \le i \le n$, each piece has exactly a_i beads of the i-th color.

For example, if $k = 2$, and the necklace is

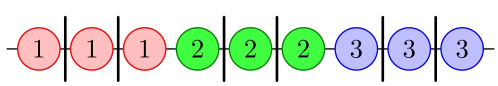


Then a possible splitting is



and then we can reassemble them to

We note that $n(k-1)$ cuts are tight for all n and k. In particular, if we arrange the necklace so that the beads of the same colors are together. Then we need at least $k-1$ cuts within each color, which implies that we need at least $n(k-1)$ cuts. The following is an example when $n = k = 3$, which also shows the optimal 6 cuts.



The proof of this theorem unexpectedly uses a lot of algebraic topology, and there is no known combinatorial proof. The case where $k = 2$ is proved by Alon and West [AW86] using Borsuk-Ulam theorem. Using an appropriate generalization of Borsuk-Ulam theorem by Bárány, Shlosman, and Szűcs [BSS81], Alon [Alo87] proved this theorem for general k. This result is also popularized in the a 3Blue1Brown video (for $k = 2$) and the a NumberPhile video.

The proof that we will give is slightly simplified from the original sources. We will need a formidable amount of algebraic topology machinery, specifically homology (Chapter 2 in [Hat02] suffices) and the general Hurewicz's theorem.

The article is organized as follows: first in Section 2, we present two quick reductions into the continuous variant and in the case where we are splitting into prime number of pieces. In Section 3, we prove the $k = 2$ case using the classical Borsuk-Ulam theorem. This proof is ultimately not necessary for the general case, but we choose to present it since it is significantly less technical. Section 4 and 5 are the algebraic topology heart of this article, in which we prove the appropriate generalization of Borsuk-Ulam theorem. Finally, Section 6 presents the main proof.

§2 Two Elementary Reductions

We now present two quick reductions.

§2.1 The Continuous Version

Theorem 2 (Necklace splitting, continuous version)**.**

Suppose that an interval $[0, 1]$ is divided into finitely many intervals, each is colored in one of the n different colors. Suppose that for all $1 \le i \le n$, the total length of intervals with color i is $k\ell_i$. Then it is possible to split the interval $[0, 1]$ using at most $n(k-1)$ cuts and reassemble them into k pieces so that for all $1 \le i \le n$, the total length of intervals with color i in each piece is ℓ_i.

▶ *Proof of Theorem 1 assuming Theorem 2.* Assume that Theorem 2. Given a configuration of necklace with $k(a_1 + \cdots + a_n) = ks$ beads, we color the interval $\left[\frac{i-1}{ks}, \frac{i}{ks}\right]$ according to the color of the i-th bead. We can now split the interval $[0, 1]$ using $n(k-1)$ cuts and reassemble them into k pieces, except that the cut may go through the middle of some bead, gaining a fractional amount of it.

To fix fractional cuts (or bad cuts), we consider a multigraph G whose vertices are pieces among $1, 2, \ldots, k$ that has at least one bad cut. For each bad cut, if the portion to the left and right ends up at piece i and j, then we draw one edge between i and j. (Each pair of vertices may have multiple edges according to the number of bad cuts.)

Observe that each vertex must have degree at least two, since we cannot have exactly one fractional bead. Thus, the number of edges in this graph is at least the number of vertices, implying that the graph has a cycle. Along that cycle, we now slide every bad cuts in the cycle by the same amount so that at least one cut no longer becomes bad. Then we can repeat this process (creating a new multigraph each time) until all bad cuts are gone. Thus, there exists a splitting with no bad cuts. □

§2.2 Reduction to Prime Case

We note the following observation.

Proposition 3.

If Theorem 2 is true for (n, k_1) and (n, k_2), then it is true for (n, k_1k_2).

▶ *Proof.* We begin by using Theorem 2 to split and reassemble the interval into k_1 pieces, using $n(k_1-1)$ cuts so that every color is evenly distributed. For each of the resulting k_2 pieces, we apply Theorem 2 to split and reassemble each piece into k_2 smaller pieces so that every color is evenly distributed. We need $n(k_2-1)$ cuts for each piece, so we need $n \cdot k_1(k_2-1)$ cuts in total. Thus, in total, we need

$$n \cdot k_1(k_2 - 1) + n(k_1 - 1) = n(k_1k_2 - 1)$$

cuts to split into k_1k_2 pieces. □

Thus, it suffices to prove Theorem 2 in the case where $k = p$, a prime number.

It may seem that this reduction does not do anything. However, quite shockingly, the proof that we will give heavily relies on the fact that k is prime.

§3 The $k = 2$ Case

We first present the proof for $k = 2$ first because it is a lot less technical. It is not needed in what follows.

Recall the following statement of Borsuk-Ulam Theorem.

Theorem 4 (Borsuk-Ulam Theorem)**.**

Suppose that $f : S^n \to \mathbb{R}^n$ is a continous function (where S^n is the n-dimensional sphere). Then there exists a point x such that $f(x) = f(-x)$.

The proof of this theorem can be found in [Hat02, 2B.6-7]. We will not repeat the proof, although some of the main ideas are given in Section 5.

We will now use Borsuk-Ulam theorem to prove Theorem 2 for $k = 2$.

► *Proof of Theorem 2 for $k = 2$.* We encode the cuts as follows: suppose that the n cuts split the intervals into sub-intervals with lengths $\ell_0, \ell_1, \ldots, \ell_n$. We map this cut to point $(x_0, x_1, \ldots, x_n) \in S^n$, where

$$x_i = \begin{cases} +\sqrt{\ell_i} & \text{if this interval ends up in the first piece;} \\ -\sqrt{\ell_i} & \text{if this interval ends up in the second piece.} \end{cases}$$

We then form a map $f : S^n \to \mathbb{R}^n$ by letting $(x_0, x_1, \ldots, x_n)$ maps to a point whose the i-th coordinate is the total length of intervals with color i in the first piece. Observe that point $-x$ corresponds to the cut where the first and second pieces are swapped. Thus, $f(-x)$ reconds the lengths of intervals of each color in the second piece. By Borsuk-Ulam theorem, there exists point $x \in S^n$ such that $f(x) = f(-x)$, which then corresponds to a cut where both pieces have equal amount of each color. □

§4 d-connected spaces

In this section, we state some useful properties of d-connected spaces that allows us to construct maps with nice properties.

We first recall some topology definitions. Throughout this chapter, all topological spaces are assumed to be a CW complex.

A group G acts **freely** on a topological space X if for all $g \in G$, the action by g is a continuous map and each point has trivial stabilizer. We assume that X and G are nice enough that the quotient X/G is a CW complex, so lifting the CW structure of X/G gives the **equivariant CW structure** on X, in which G acts freely on the cells. This can be constructed for each specific action that we use in the proof, but we will not dwell on these minor technical details.

If G acts on both X and Y, a map $f : X \to Y$ is **G-equivariant** if $f(gx) = gf(y)$ for all $g \in G$.

For each path-connected topological space X, we let $\pi_d(X)$ denote the d-th homotopy group, which is the group consists of homotopy classes of maps $S^d \to X$. We let $H_d(X)$ denote the d-th homology group.

We now introduce the central definition of this section.

Definition 5.

A topological space is ***d*-connected** if and only if X is path-connected and $\pi_1(X) = \pi_2(X) = \cdots = \pi_d(X) = 0$.

By the general version of Hurewicz theorem [Hat02, Thm. 4.32], X is d-connected if and only if X is path-connected, $\pi_1(X) = 0$, and $H_2(X) = \cdots = H_d(X) = 0$. This gives us a way to check that a space is d-connected without having to deal with higher homotopy groups.

The d-connectedness allows us to construct maps between topological spaces with desired properties easily. We explain this in steps with increasing generality.

Lemma 6.

Let X be a topological space such that $\pi_i(X) = 0$. Then for any map $\phi : S^i \to X$, there exists a lift $\widetilde{\phi} : D^{i+1} \to X$ that commutes with the boundary inclusion.

$$\begin{tikzcd} D^{i+1} \arrow[dr, "\widetilde{\phi}"] & \\ S^i = \partial D^{i+1} \arrow[u, hook] \arrow[r, "\phi"'] & X \end{tikzcd}$$

▶ *Proof.* Since $\pi_i(X) = 0$, we have that ϕ is nulhomotopic. The homotopy gives a map $h : S^i \times [0,1] \to X$ such that $\phi(\bullet, 1)$ is a contant map. By identifying $S^i \times \{1\}$ to a point, we get a map $\widetilde{\phi} : D^i \times X$. ☐

We now use the above lemma to deduce useful results.

Proposition 7.

Let X be a d-connected topological space, and let Y be a CW complex of dimension at most $d+1$. Suppose that a finite group G acts freely on both X and Y. Then there exists a G-equivariant map $f : Y \to X$.

▶ *Proof.* We take a G-equivariant CW structure on Y. We use induction to define a map $\phi_i : \mathrm{Sk}^i\, Y \to X$. For the base case $i = 0$, we simply have to pick where each vertex is sent to. We do so in a way that it is G-equivariant.

For the inductive step, suppose we define ϕ_i. We have to define ϕ_{i+1}. The previous lemma allows us to attach the $(i+1)$-cell. We have to do it in a way that it is G-equivariant. This is easy, since for each cell we attach, we attach all of its G-orbit in a way that it is equivariant. This completes the proof. ☐

Proposition 8.

Let X be a d-connected topological space, and let Y be a CW complex of dimension at most d. Then any two maps $f, g : Y \to X$ are homotopic.

Furthermore, if a finite group G acts freely on both X and Y, then the homotopy can be made equivariant.

▶ *Proof.* We use induction to define a homotopy $h_i : \mathrm{Sk}^i\, Y \times [0,1] \to X$. The base case $i = 0$ is clear since X is path-connected. For the inductive step, suppose that h_i is defined for some $i = 0, 1, \ldots, d-1$. We have to attach $(i+1)$-cells to define h_{i+1}. More specifically, we have to lift a map $\phi : S^i \times [0,1] \to X$ to a map $\widetilde{\phi} : D^{i+1} \times [0,1] \to X$ that commutes with boundary inclusion and its restrictions onto $D^{i+1} \times \{0\}$ and $D^{i+1} \times \{1\}$ coincide with f and g. This will complete the proof since we can use this to attach an $(i+1)$-cell. This process can be made equivariant if we take a G-equivariant CW structure on Y.

Note that $D^{i+1} \times [0,1] \simeq D^{i+2}$, and the boundary S^{i+1} is the union of $S^i \times [0,1]$, $D^{i+1} \times \{0\}$, and $D^{i+1} \times \{1\}$. In particular, we have the map $S^{i+1} \to X$ defined where the piece $S^i \times [0,1]$ comes from ϕ, the piece $D^{i+1} \times \{0\}$ comes from f, and the piece $D^{i+1} \times \{1\}$ comes from g. By the previous lemma, this map lifts to the $D^{i+2} = D^{i+1} \times [0,1] \to X$ as desired. □

§5 Generalization of Borsuk-Ulam Theorem

We now state the generalization of Borsuk-Ulam theorem that we will use.

We let μ_p be the group of the p-th root of unity. This is isomorphic to $\mathbb{Z}/p$, but it will be more natural to write the group with multiplicative notation.

Theorem 9.

Let p be a prime number and n be a positive integer. Suppose that X is an $(n(p-1)-1)$-connected space such that μ_p acts freely. Let $f : X \to \mathbb{R}^n$ be a map. Then there exists $x \in X$ such that $f(x) = f(\omega x) = f(\omega^2 x) = \dots$, where ω is a generator of μ_p.

The proof of this mirrors the proof of Borsuk-Ulam theorem. In particular, the key lemma there is the **odd degree theorem** that if $f : S^n \to S^n$ is an odd map, then $\deg f$ (defined as the action of f on H_n) is odd. The following proposition is an analogue of the odd degree theorem.

Proposition 10.

If μ_p acts freely on S^n and a map $f : S^n \to S^n$ is μ_p-equivariant, then $\deg f \equiv 1 \pmod p$.

► *Proof.* Unfortunately, the proof for $p = 2$ in [Hat02] does not quite work here because we no longer have the transfer sequence. We will have to use a slightly different argument that requires some homotopy theory.

Consider a μ_p-equivariant cellular structure of S^n (in particular, μ_p acts freely on the cells), and let C_* be the corresponding cellular chain complex. We recall that the homology of this chain complex is $H_0(S^n, \mathbb{F}_p) = H_n(S^n, \mathbb{F}_p) = \mathbb{F}_p$ but $H_k(S^n, \mathbb{F}_p) = 0$ for all $k \neq 0, n$. We let $z \in C_n$ be the generator of $H_n(S^n, \mathbb{F}_p)$ (which is well-defined since C_n has no boundary).

By Proposition 8, the restrictions of f and id onto $\operatorname{Sk}_{n-1} S^n$ are equivariantly homotopic. By homotopy extension property, we thus can replace f with an equivariantly homotopic map so that $f = \mathrm{id}$ on $\operatorname{Sk}_{n-1} S^n$. In particular, since C_* is defined as relative homology over skeleton, we get that $f_* : C_* \to C_*$ is well-defined.

For any cell $e \in \operatorname{Sk}_n S^n$, we have

$$\partial(f_*(e) - e) = f_*(\partial e) - \partial e = 0,$$

since $f = \mathrm{id}$ on the $(n-1)$-skeleton. Thus, $f_*(e) - e = kz$ for some $k \in \mathbb{F}_p$.

However, we observe that if ω is a generator of μ_p, then $\omega_* z = z$. This is because if $\omega_* z = az$ for some $a \in \mathbb{F}_p$, then $a^p = 1$, so $a = 1$. In particular, z is a sum of μ_p-orbit of cells. If $e, \omega e, \omega^2 e, \dots, \omega^{p-1} e$ is one orbit and $s = e + \omega e + \omega^2 e + \dots + \omega^{p-1} e$, then

$$f_*(s) - s = k(z + \omega_* z + \omega_* z^2 + \dots + \omega_*^{p-1} z) = k \cdot pz = 0.$$

Since z is a sum of elements of the form s, it follows that $f_*(z) - z = 0$, so $\deg f \equiv 1 \pmod p$. □

▶ *Proof of Theorem 9.* We define the map $F : X \to (\mathbb{R}^n)^p$ by $F(x) = (f(x), f(\omega x), \ldots, f(\omega^{p-1}x))$. Assume for contradiction that there does not exist such X. Then F restricts to the map $X \to (\mathbb{R}^n)^p \setminus D$, where D is the diagonal. We let

$$D^\perp = \left\{ (a_{ij})_{1\le i\le n, 1\le j\le p} : \sum_{j=1}^{p} a_{ij} = 0 \text{ for all } i \right\}$$

be the orthogonal complement of D. We then consider the map $\phi : X \to S^N$ defined by the composition

$$X \xrightarrow{F} (\mathbb{R}^n)^p \setminus D \longrightarrow D^\perp \setminus \{0\} \simeq \mathbb{R}^{n(p-1)} \setminus \{0\} \longrightarrow S^{n(p-1)-1},$$

where the second map is projection onto subspace and the third map is radial projection.

The group μ_p acts on $(\mathbb{R}^n)^p$ by rotating each block of p coordinates, so it acts on $D^\perp$, $\mathbb{R}^{n(p-1)}$, and $S^{n(p-1)-1}$ in the same way. Therefore, ϕ is μ_p-equivariant.

Since X is $(n(p-1)-1)$-connected, by Proposition 7 there exists a μ_p-equivariant map $S^{n(p-1)-1} \to X$. Composing with ϕ gives a μ_p-equivariant map $S^{n(p-1)-1} \to S^{n(p-1)-1}$. Since $H_{n(p-1)-1}(X) = 0$ (because X is $(n(p-1)-1)$-connected), we get that this map has degree 0. This contradicts Proposition 10. □

§6 Proof for $k = p$

We now prove Theorem 2 in the case of $k = p$.

Like the case $k = 2$, we need some way to model the cuts. Let $N = n(p-1)$ be the number of cuts. We let U be the quotient

$$U = \frac{\mu_p \times [0,1]}{(\omega, 0) \sim (\omega', 0) \text{ for all } \omega, \omega' \in \mu_p}$$

and have μ_p acts on by the first coordinate: $\omega \cdot (\omega', x) = (\omega\omega', x)$. We define

$$X_{N,p} = \{(\omega_i, x_i)_{i=1}^{N+1} \in U^{N+1} : x_1 + x_2 + \cdots + x_{N+1} = 1\}.$$

Intuitively, a_i denote the size of i-th cut and ω_i denote which piece that the i-th cut is assigned to. It does not matter where cuts of size 0 is assigned, which explains why we are taking quotient in the definition of U. This quotienting makes $X_{N,p}$ path-connected and has interesting topology that we will be able to utilize topological arguments.

We define the map

$$\begin{aligned} f : X_{N,p} &\to \mathbb{R}^n \\ (\omega_i, x_i)_{i=1}^{N+1} &\mapsto \left(\sum_{i:\omega_i=1} \begin{pmatrix} \text{total length of color } c \\ \text{in } [s_{i-1}, s_i] \end{pmatrix} \right)_{c=1}^{n} \end{aligned}$$

where $s_i = x_1 + \cdots + x_i$. Thus, $f(x)$ counts the total length of beads in each color for the first piece. Similarly, $f(\omega x), f(\omega^2 x), \ldots$ counts the total length of beads in each color for other pieces. Therefore, we may apply Theorem 9 if we prove the following.

📝 **Proposition 11.**

$X_{N,p}$ is $(N-1)$-connected.

▶ *Proof.* This is a pure topology calculation, and there are many ways to approach this. We first show that the fundamental group $\pi_1(X_{N,p})$ is trivial, which will show that $H_1(X_{N,p}) = 0$. To do this, we use van-Kampen theorem on the sets

$$U_t = \left\{(\omega_i, x_i)_{i=1}^{N+1} \in X_{N,p} : \omega_{N+1} = t\right\}$$

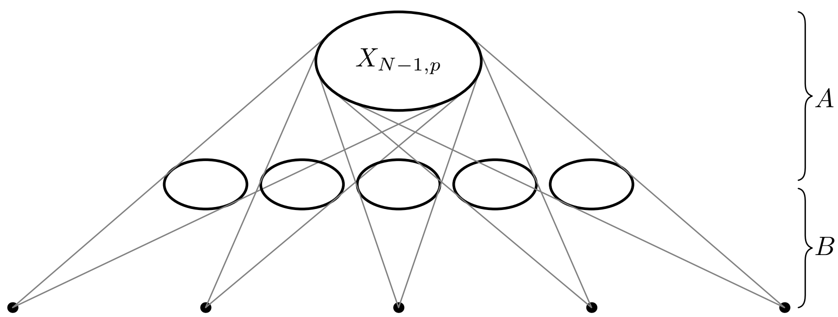
for $t \in \mu_p$. (In the sketch below, U_t is each cone.)

We have that $U_{t_1} \cap U_{t_2} = X_{N-1,p}$ for all $t_1 \neq t_2$, which implies that the double and triple intersections are path-connected. Since U_t deformation retracts to a point by increasing the x_{N+1} coordinate to 1, it follows that $\pi_1(X_{N,p})$ is trivial.

We now compute the homology groups and show that $H_k(X_{N,p}) = 0$ for all $k = 2, \ldots, N-1$ by induction on N (we use reduced homology). The base case $N = 0$ is vacuous. For the inductive step, we decompose $X_{N,p}$ into two parts:

$$A = \left\{(\omega_i, x_i)_{i=1}^{N+1} \in X_{N,p} : 0 \leq x_{N+1} < \tfrac{2}{3}\right\}$$
$$B = \left\{(\omega_i, x_i)_{i=1}^{N+1} \in X_{N,p} : \tfrac{1}{3} < x_{N+1} \leq 1\right\}.$$

Here is the rough sketch of the space $X_{N,p}$. The top corresponds to $x_{N+1} = 0$, while the bottom corresponds to $x_{N+1} = 1$.



We note that

- A deformation retracts to $X_{N-1,p}$ by shrinking the x_{N+1}-coordinate to 0.
- B deformation retracts to a disjoint union of p points by increasing the x_{N+1}-coordinate to 1.
- $A \cap B$ is homeomorphic to a disjoint union of p copies of $X_{N-1,p}$.
- The inclusion $A \hookrightarrow X_{N,p}$ is nulhomotopic by pushing everything towards one vertex in B.

The Mayer-Vietoris sequence reads

$$H_k(A) \oplus H_k(B) \longrightarrow H_k(X_{N,p}) \xrightarrow{\delta} H_{k-1}(A \cap B) \longrightarrow H_{k-1}(A) \oplus H_{k-1}(B).$$

If $k = 2, 3, \ldots, N-1$, then $H_{k-1}(X_{N-1,p}) = 0$ the first map is a zero map, so we actually have

$$0 \longrightarrow H_k(X_{N,p}) \longrightarrow H_k(A \cap B) = H_{k-1}(X_{N-1,p})^{\oplus p} = 0,$$

so we have that $H_k(X_{N,p}) = 0$. □

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