Algebraic Topology

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Preface

▶ **What is Algebraic Topology?** Informally, topology is a study of shapes and their continuous deformations. For example, a square is considered the same as a circle, since we can continuously deform one to another. The ultimate goal of topology is to classify spaces up to continuous deformation. In doing so, we need to be able to distinguish two topological spaces and show that there is no continuous deformation between the two.

Algebraic topology provides a set of useful tools to classify spaces. In particular, it provides a set of **algebraic invariants** that do not change under continuous deformation. Two most common invariants that will form a central core of this book are

- **homotopy groups** $\pi_n(X)$ where $n = 1, 2, \ldots$, which is defined as the group of maps $S^n \to X$; and modulo continuous deformation;
- **homology groups** $H_n(X)$ where $n = 1, 2, \ldots$, which is an abelian group generated by n-dimensional holes of X.

Homotopy groups are simpler to define. The first homotopy group $\pi_1(X)$ is known as the **fundamental group**, which is also easy to compute. Therefore, this will be our first invariant that we study. However, higher homotopy groups $\pi_2(X), \pi_3(X), \ldots$ are difficult to compute, and (for now) we do not discuss them in this book.

In contrast, the homology groups $H_n(X)$ are much more technical but more straightforward to compute even for higher n. The algebraic machinery of homology has now become a separate branch of mathematics called homological algebra, which is a powerful algebraic tool that has found applications in commutative algebra, number theory, and algebraic geometry.

More broadly, algebraic topology has become a fundamental discipline in mathematics, and it tools have been used everywhere in mathematics, particularly in algebraic geometry, differential geometry, and category theory. This book aim to provide an introduction to this subject.

▶ **Prerequisite** We assume that the reader is familiar with groups, rings, and modules. Chapter A provides necessary background in category theory, which the reader can read as they go along in the text. Although we do not require any topology background, familiarity with the concepts of open and closed set (e.g., of the real line or a metric space) is helpful in the first chapter.

▶ **Content** These notes cover material in standard algebraic topology courses. At MIT, this correspond to two courses: 18.901 (Introduction to Topology) which roughly covers Chapter 1-3, and 18.905 (Algebraic Topology I) which roughly covers Chapter 4-8.

- **Chapter 1 (Topological Spaces).** Before we can distinguish spaces, we need to precisely define a notion of spaces. This chapter introduces the definition of topological spaces, common topological constructions, and some useful notions such as connectedness or compactness.
- **Chapter 2 (Fundamental Group).** This chapter introduces the first invariant, the fundamental group $\pi_1(X)$, which is a group of loops in X modulo continuous deformation. In the process, we explain the concept of homotopy and homotopy equivalence. We state and sketch the proof of van Kampen theorem, a theorem that allows us to compute the fundamental group of a topological space by breaking it up into simpler pieces.

- **Chapter 3 (Covering Spaces).** In this chapter, we discuss the theory of covering spaces, which is a type of objects that we know how to classify completely algebraically in terms of π_1. This gives a geometric interpretation of π_1, which can be used to prove some algebraic results that does not seem to involve topology at all.
- **Chapter 4-5 (Homology).** This chapter introduces a new invariant $H_n(X)$, which is based on the n-dimensional holes of X. Unfortunately, setting up basic properties of this invariant occupies most of Chapter 4. In Chapter 5, we discuss techniques for computing homology, including Mayer-Vietoris sequence and cellular homology.
- **Chapter 6 (Homology with Coefficients).** This chapter introduces homology, but the coefficients can be arbitrary ring. We introduce a method to take a product of two chain complexes and explain how to deduce homology with coefficients from (integral) homology. We also give a recipe for computing homology of product of two spaces.
- **Chapter 7 (Cohomology).** This chapter introduces another invariant called cohomology groups $H^n(X)$. Roughly speaking, cohomology is an invariant that dualizes homology (i.e., reverse the direction of maps). The groups $H^n(X)$ alone do not give additional information, but the important upshot is we will be able to turn $H^*(X)$ into a ring. This ring provides additional invariant.
- **Chapter 8 (Poincaré Duality).** In this chapter, we specialize to n-manifolds and explores the symmetry between k-dimensional objects and $(n-k)$-dimensional objects known as Poincaŕe duality. The huge payoff is a quick computation of cohomology rings of various spaces. We close this chapter by also introducing intersection pairing and Lefschetz Fixed Point Theorem, which counts the number of fixed points of a map purely algebraically.

All contents in this book existed in some form. The two standard textbooks that we refer to are Hatcher [Hat02] and Miller [Mil21]. In general, we put more categorial/algebraic spin than [Hat02] and thus following [Mil21] more closely in the second half of the text. In general, we are not afraid to introduce more sophisticated tools when it gives a conceptually simpler and memorable proofs, such as the proof of Seifert-van Kampen theorem as given in Section 3.3.2, the use of naturality in the proof of homotopy invariance of homology in Theorem 4.2.8, or the introduction of acyclic models in Section 6.6.

► **History and Acknowledgements** These notes were based on the lecture notes when the author took 18.901 (Introduction to Topology) taught by Anthony Conway in Spring 2023 and 18.905 (Algebraic Topology I) taught by Jeremy Hahn in Fall 2024, both at MIT. Since then, parts of these notes have been rewritten and/or reorganized to form a coherent set of notes. The author thanks both professor for teaching the class; their insights and presentation of materials have been imparted throughout this book.

Contents

1 Topological Spaces

Informally, topology is a study of spaces (or shapes) under continuous deformations. Before we can study any geometrical spaces, we need to have a mathematical object that models those spaces. Historically, people have been viewing spaces as a subset of common spaces such as $\mathbb{R}^n$. For example, a sphere S^2 can be written a subset of $\mathbb{R}^3$ as

$$\left\{(x, y, z) : x^2 + y^2 + z^2 = 1\right\},$$

or a torus can be written as a subset of $\mathbb{R}^3$ by

$$\left\{(x, y, z) : \left(\sqrt{x^2 + y^2} - 2\right)^2 + z^2 = 1\right\} \subseteq \mathbb{R}^3.$$

However, as spaces get more complicated, writing equations like the one above is cumbersome. The goal of this chapter is to move away from concrete equations to abstract spaces, known as **topological spaces**. Topological spaces contain less information, but the benefit is that to describe a new topological space, we need to input less information to it, making constructing complicated spaces convenient.

§1.1 Topological Spaces and Continuous Maps

In this section, we precisely define topological spaces and what it means for a map to be continuous under this new kind of spaces.

§1.1.1 Topological Spaces

Such mathematical object is a topological space, defined as the following.

Definition 1.1.1 (Topological Spaces).

A **topological space** consists of

- a set X of points; and
- a collection $\mathcal{T}$ of subsets of X

such that

- $\emptyset$ and X are in $\mathcal{T}$;
- $\mathcal{T}$ is closed under arbitrary union: if $(U_i)_{i \in I}$ is a (possibly infinite, even uncountably infinite) collection of sets such that $U_i \in \mathcal{T}$, then the union $\bigcup_{i \in I} U_i \in \mathcal{T}$; and
- $\mathcal{T}$ is closed under finite intersection: if $U, V \in \mathcal{T}$, then $U \cap V \in \mathcal{T}$.

A set is said to be **open** if it is in $\mathcal{T}$. In other words, $\mathcal{T}$ is a collection of open sets.

At a glance, this seems to have nothing to do with geometry. However, the actual point of this definition is that **the list of open sets is all the information we need**. In particular, to construct a topological space, we only need to give a list of open sets. In the next section, we will see how this allows for easier construction of familiar spaces such as a torus, or a Klein bottle, without having to write down a single equation.

Example 1.1.2 (Discrete Topology).

Given any set X, a **discrete topology** on set X is a topology where every set is open. Intuitively, this topology looks like a disjoint union of points.

Example 1.1.3 ($\mathbb{R}$ and $\mathbb{R}^n$).

The **standard topology** on $\mathbb{R}$ is defined consistently with the usual definition of open sets: a set $U \subseteq \mathbb{R}$ is open if and only if for any $p \in U$, there exists $\varepsilon > 0$ such that the open ball

$$B_\varepsilon(p) := \{x \in \mathbb{R} : |x - p| < \varepsilon\}$$

is contained in U. For example, union of open intervals like $(2, 3) \cup (4, 5)$ is open.

Let us take time to check that this topology satisfies the axioms of topological spaces.

- First, clearly both $\emptyset$ and $\mathbb{R}$ are open.
- Given a collection of open sets $(U_i)_{i\in I}$ and a point $p \in \bigcup_{i\in I} U_i$, we have to show that an open ball around p is contained in the union. To do this, note that $p \in U_i$ for some $i \in I$, so there exists $\varepsilon > 0$ such that $B_\varepsilon(p) \subseteq U_i$, which in turn is contained in the union $\bigcup_{i\in I} U_i$, so $B_\varepsilon(p) \subseteq \bigcup_{i\in I} U_i$ as desired.
- Given open sets U and V, we have to show that $U \cap V$ is open. Fix any point $p \in U \cap V$. Then there exists $\varepsilon_1 > 0$ such that $B_{\varepsilon_1}(p) \subseteq U$ and $\varepsilon_2 > 0$ such that $B_{\varepsilon_2}(p) \subseteq V$. Then $B_{\min(\varepsilon_1,\varepsilon_2)}(p) \subseteq U \cap V$, implying that there is a ball contained in $U \cap V$.

  Notice that this does not work for infinite intersection.

Hence, the topology described above is a valid topological space.

More generally, we can define a standard topology on $\mathbb{R}^n$ by a set $U \subseteq \mathbb{R}^n$ is open if and only if for any $p \in U$, there exists $\varepsilon > 0$ such that the open ball

$$B_\varepsilon(p) := \{x \in \mathbb{R}^n : |x - p| < \varepsilon\}$$

is contained in U.

(Recall that if $x = (x_1, \dots, x_n)$, then $|x| = \sqrt{x_1^2 + \cdots + x_n^2}$.)

Example 1.1.4 (Subspace topology).

Given a topological space X and a subset $A \subseteq X$, we define the **subspace topology** on A by $U \subseteq A$ is open if and only if it is of the form $U = V \cap A$ for some open set $V \subseteq X$.

With this, we can now define various spaces. For example, a **sphere** S^n and a **disk** D^{n+1} are subspaces of $\mathbb{R}^{n+1}$

$$\begin{aligned} S^n &= \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} : x_1^2 + \cdots + x_{n+1}^2 = 1\} \\ D^{n+1} &= \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} : x_1^2 + \cdots + x_{n+1}^2 \leq 1\}. \end{aligned}$$

This is not the best definition to take because it relies on equations. Soon, we will learn better ways to construct a sphere.

Our next task is to recover some basic definitions that we have seen on $\mathbb{R}^n$ based on open sets alone. First, closed sets have an easy relation with open sets.

Definition 1.1.5.

Given a topology on set X, a subset $C \subseteq X$ is **closed** if and only if its complement $X \setminus C$ is open.

Example 1.1.6.

In a standard topology on $\mathbb{R}$, $[2,3]$ is closed because $\mathbb{R} \setminus [2,3] = (-\infty, 2) \cup (3, \infty)$, which is open because it is a union of two open intervals.

§1.1.2 Continuous Maps

Definition 1.1.7.

Given topological spaces X and Y, a function $f : X \to Y$ is **continuous** if and only if for any open set $V \subseteq Y$, the preimage $f^{-1}V$ is an open set in X.

This definition is not very intuitive, so let us convince ourselves that this coincides with a normal definition in $\mathbb{R}$.

Proposition 1.1.8.

A function $f : \mathbb{R} \to \mathbb{R}$ is continuous if and only if for every $a \in \mathbb{R}$ and $\varepsilon > 0$, there exists δ (which can depend on a and ε) such that if $|x - a| < \delta$, then $|f(x) - f(a)| < \varepsilon$.

▶ *Proof.* ($\Rightarrow$) Suppose that f is continuous. Then the preimage of $B_\varepsilon(f(a))$ must be an open set that contains a. In particular, there exists δ such that $B_\delta(a) \subseteq f^{-1}B_\varepsilon(f(a))$. For any x such that $|x - a| < \delta$, we have $x \in f^{-1}B_\varepsilon(f(a))$, so $f(x) \in B_\varepsilon(f(a))$, or $|f(x) - f(a)| < \varepsilon$, as desired.

($\Leftarrow$) Now, suppose that the δ-ε condition holds. Let U be an open set of $\mathbb{R}$, and let $a \in f^{-1}U$. Therefore, $f(a) \in U$, so there exists $\varepsilon > 0$ such that $B_\varepsilon(f(a)) \subseteq U$. Now, there exists $\delta > 0$ such that if $|x - a| < \delta$ then $|f(x) - f(a)| < \varepsilon$.

We claim that $B_\delta(a) \subseteq f^{-1}U$. To see this, note that if $x \in B_\delta(a)$, or $|x - a| < \delta$, then $|f(x) - f(a)| < \varepsilon$, so $f(x) \in B_\varepsilon(f(a))$, which means that $f(x) \in U$, so $x \in f^{-1}U$, as desired. □

Similar arguments hold for $\mathbb{R}^n$, so we can use the standard definition there as well.

Exercise 1.1.9. Prove that if $f : X \to Y$ and $g : Y \to Z$ are continuous, then so is $f \circ g$.

Definition 1.1.10.

The category **Top** of topological spaces is the category whose objects are topological spaces and morphisms are continuous maps.

§1.1.3 Homeomorphism

Informally, two topological spaces are homeomorphic if they are related by a continuous deformation. Thus, topologists are interested in studying topological spaces up to homeomorphism. The following definition makes this precise.

Definition 1.1.11.

Two topological spaces X and Y are **homeomorphic** if and only if there exists continuous maps $f : X \to Y$ and $g : Y \to X$ such that $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$.

Maps f and g are called the **homeomorphism**.

Warning 1.1.12.

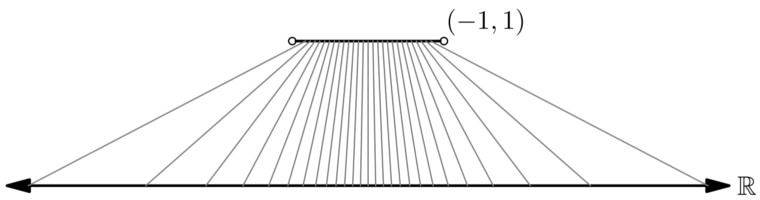
Having a continuous bijection $f : X \to Y$ is not enough to be homeomorphic. (Can you find a counterexample? There is one with $|X| = |Y| = 2$.) We also need f^{-1} to be continuous as well.

Example 1.1.13.

The interval $(-1, 1)$ and $\mathbb{R}$ are homeomorphic, and one possible homeomorphism is given by

$$f : (-1,1) \to \mathbb{R} \qquad\qquad g : \mathbb{R} \to (-1,1)$$
$$x \mapsto \tan(\pi x/2)\,, \qquad\qquad x \mapsto \tfrac{2}{\pi}\arctan x\,.$$

In picture, the map looks like the following:



More generally, any open interval is homeomorphic to $\mathbb{R}$.

Example 1.1.14.

We have that $S^n \setminus \{\text{point}\}$ (sphere S^n with one point removed) is homeomorphic to $\mathbb{R}^n$. We leave the detail to Problem 1.A.

§1.2 Constructions

In this section, we give constructions of various common topological spaces.

§1.2.1 Product and Disjoint Union

► **Products.** Given topological spaces X and Y, we explain how to construct the product topology $X \times Y$.

Definition 1.2.1.

Given a topological spaces X and Y, we endow the **product topology** on $X \times Y$ as follows: a subset of $X \times Y$ is open if and only if it is a **union** of sets of the form $U \times V$ where $U \subseteq X$ and $V \subseteq Y$ are both open.

Similarly, given topological spaces $X_1, \dots, X_n$, the open sets of $X_1 \times \dots \times X_n$ are unions of sets of the form $U_1 \times \dots \times U_n$, where $U_i \subseteq X_i$ are open.

Warning 1.2.2.

For infinite product such as $X_1 \times X_2 \times \dots$, the open sets of $X_1 \times X_2 \times \dots$ are **not** union of sets of the form $U_1 \times U_2 \times \dots$. (One require $U_i = X_i$ for all but finitely many i's.) We will not consider infinite products in this book.

If you know category theory, the following proposition makes the product a valid product.

Proposition 1.2.3.

(a) The projection maps $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$, defined by projection to first and second coordinate respectively, are both continuous.

(b) For any topological space Z and continuous maps $f : Z \to X$ and $g : Z \to Y$

$$\begin{tikzcd}
Z \arrow[dr, dashed, "\exists! h"] \arrow[drr, bend left, "\pi_X"] \arrow[ddr, bend right, "\pi_Y"'] & & \\
& X \times Y \arrow[r, "f"'] \arrow[d, "g"] & X \\
& Y &
\end{tikzcd}$$

there exists unique continuous map $h : Z \to X \times Y$ that makes the above diagram commutes (i.e., $\pi_Y = g \circ h$ and $\pi_X = f \circ h$).

▶ *Proof.* (a) The preimage of $\pi_X^{-1} U$ is $U \times Y$, which is clearly open.

(b) The map h is given by $h(z) = (f(z), g(z))$. It suffices to show that this map is continuous. To do that, note that since preimage behaves well under union, it suffices to show that $h^{-1}(U \times V)$ is open, but this set is $f^{-1}U \times g^{-1}V$, which is open, so we are done. □

Example 1.2.4.

- The product $[0, 1] \times S^1$ is a cylinder.
- The product $S^1 \times S^1$ is a torus. Can you visualize why this is true?

▶ Disjoint Union.

Definition 1.2.5.

Given two topological spaces X and Y, their **disjoint union** $X \amalg Y$ is a topological space such that

- the set of points consists of a copy of X and a copy of Y (one way to write this formally is $X \times \{0\} \cup Y \times \{1\}$);
- the open sets are an open set of X union with an open set of Y.

(Similarly, we can write infinite disjoint union $\coprod_{i \in I} X_i$, and the open sets are union of open sets $U_i \subseteq X_i$.)

Disjoint union of topological spaces is also known as the **coproduct**. This is because it satisfies the same universal property as product, but all arrows are reversed. In particular, the inclusion maps $\iota_X : X \to X \amalg Y$ and $\iota_Y : Y \to X \amalg Y$ are continuous. Moreover, for any topological space Z and continuous maps $f : X \to Z$ and $g : Y \to Z$, there exists unique $h : X \amalg Y \to Z$ such that $h \circ \iota_X = f$ and $h \circ \iota_Y = g$.

§1.2.2 Quotient

Given a topological space X, we may identify some points to be the same, giving a new topological space. This construction is called the **quotient topology** and it shows how only needing to specify open sets makes it much easier to construct a new topological space.

Definition 1.2.6.

Given a topological space X and an equivalence relation $\sim$ on set of points, we define the **quotient topology** $X/\sim$ by

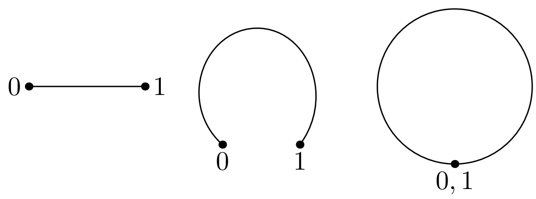
- the set of points being the set of equivalence classes;
- if $\pi : X \to X/\sim$ is the projection map (map x to the equivalence class $[x]$ containing x), then a set $U \subseteq X/\sim$ is open if and only if $\pi^{-1}U = \{x : [x] \in U\}$ is open in X.

Notice that this definition is rigged so that the projection map $\pi : X \to X/\sim$ is continuous.

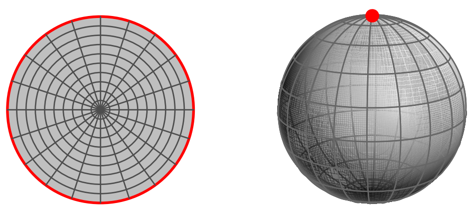
Let's see some examples.

Example 1.2.7 (Sphere).

- Suppose that we consider $[0,1]$ and define an equivalence relation $\sim$ that identifies the endpoints (formally, $x \sim y$ if and only if $x = y$ or $\{x, y\} = \{0, 1\}$), then $[0,1]/\sim$ is a sphere S^1.



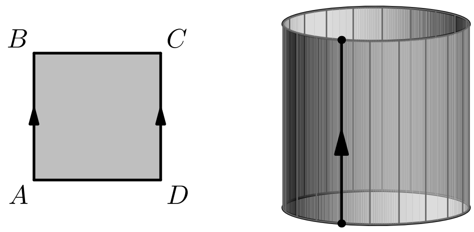
- More generally, suppose that we have D^n and define an equivalence relation $\sim$ that identifies the boundary S^{n-1} to a single point, then $D^n/\sim \simeq S^n$. An example when $n = 2$ is shown below.



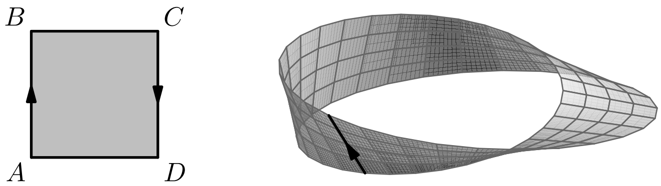
Example 1.2.8 (Cylinder, Möbius strip, torus, and Klein bottle).

Consider a square $ABCD$ (homeomorphic to $[0,1]^2$).

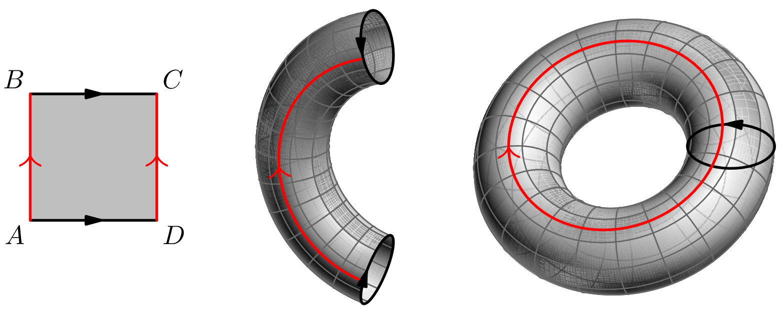
- If we identify edge AB with edge DC, we get a **cylinder** $[0,1] \times S^1$.



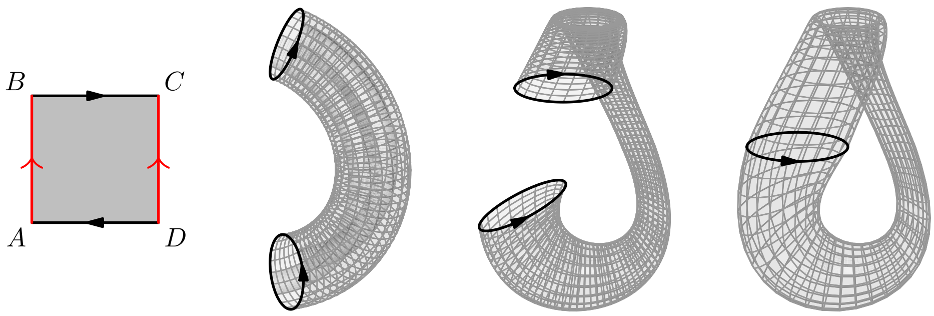
- If we identify edge AB with edge CD, we get the **Möbius strip**.



- If we identify edges AB and DC together and edges BC and AD together, then we get a **torus** $S^1 \times S^1$.

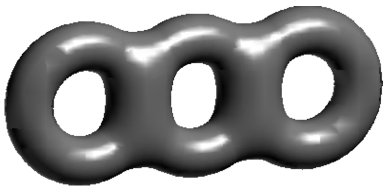


- If we identify edges AB and DC together and edges BC and DA together, then we get a **Klein bottle**.

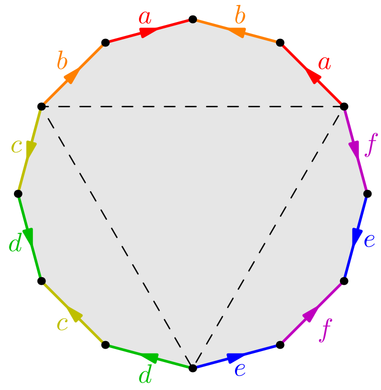


Example 1.2.9 (g-holed torus)**.**

A **g-holed torus** Σ_g can be realized as a quotient of a $4g$-gon by identifying some edges together. For example, if $g = 3$, then Σ_3 is shown below.



It is obtained by identifying the two edges labeled a together, the two edges labeled b together, and so on.



In general, Σ_g is a quotient of a $4g$-gon with edges labeled in a similar fashion as above.

Seeing why the above diagram gives the desired 3-holed torus require quite a bit of visualization that is best shown on a video rather than on a paper. I recommend `https://www.youtube.com/watch?v=U5N5mg3MePM`.

We note the following important special cases of quotient topology.

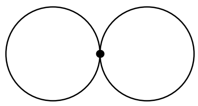
Definition 1.2.10 (Wedge Sums)**.**

Given two topological spaces X and Y with points $x_0 \in X$ and $y_0 \in Y$, their **wedge sum** $X \vee Y$ is the quotient

$$\frac{X \coprod Y}{\{x_0 \sim y_0\}}$$

modulo only one relation $x_0 \sim y_0$.

Usually, we omit the basepoint if it does not matter. For example, the following diagram is $S^1 \vee S^1$.



Definition 1.2.11.

If A is a subset of X, then X/A is a topological space obtained from X by identifying every point in A to a single point.

§1.2.3 CW Complexes

Pretty much all topological spaces you will care about in algebraic topology is a CW complexes, which is a space that is obtained by

- starting from a bunch of points (i.e., 0-cells);
- attaching some lines D^1 (i.e., 1-cells) connecting some points;
- attaching some disks D^2 (i.e., 2-cells);
- $\vdots$

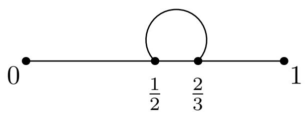
We now define those rigorously. First, let us define what attaching means.

Definition 1.2.12 (Cell attachment).

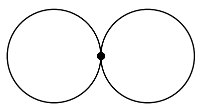
Given a topological space X, one can construct a new topological space X' by **attaching a k-cell** D^k to X. The data required for this process is the **attaching map** $f : \partial D^k \to X$. The new space X' is obtained by taking a disjoint union of X and D^k, identifying the boundary ∂D^k with the map f. Explicitly, the new space X' is defined by

$$X' = \frac{X \amalg D^k}{\langle p \sim f(p) \text{ for all } p \in \partial D^k \subset D^k \rangle}.$$

Example 1.2.13 (Examples of Cell Attachments).



If $B = [0,1]$ and $f : S^0 \to [0,1]$ with image $\{\frac{1}{2}, \frac{2}{3}\}$, then D will be interval $[0,1]$ but we attach a segment from $\frac{1}{2}$ to $\frac{2}{3}$.



The $S^1 \vee S^1$ is obtained by attaching two 1-cells to a point.

We now formally define a CW Complex.

Definition 1.2.14 (CW Complex).

A **CW Complex** is a space X with a filtration

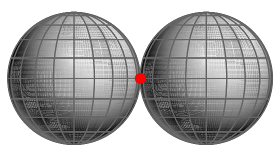
$$\emptyset \subset \mathrm{Sk}_0 X \subset \mathrm{Sk}_1 X \subset \mathrm{Sk}_2 X \subset \cdots \subset X,$$

such that

- $\bigcup_{n \in \mathbb{N}} \mathrm{Sk}_n X = X$
- $\mathrm{Sk}_{n+1} X$ is obtained from $\mathrm{Sk}_n X$ by attaching (possibly many or no) $(n+1)$-cells.

Example 1.2.15 ($S^2 \vee S^2$).

Let $X = S^2 \vee S^2$ be two spheres kissing at a point.



Then X can be realized as a CW complex in at least two different ways:

- $\mathrm{Sk}_0\, X$ is a point, $\mathrm{Sk}_1\, X$ is still a point, and $\mathrm{Sk}_2\, X$ is obtianed by attaching two discs around that point. More formally, the attaching map is sending every point in the boundary to a single point.
- $\mathrm{Sk}_1\, X = S^1 \vee S^1$, and then attach four discs, two for each loop in $\mathrm{Sk}_2\, X$.

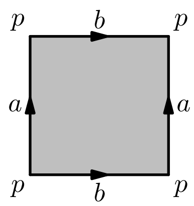
Example 1.2.16 (Torus is a CW Complex)**.**

A torus is a CW complex with

$$\mathrm{Sk}_0\, X = \text{point}, \quad \mathrm{Sk}_1\, X = \text{figure-8}, \quad \mathrm{Sk}_2\, X = X.$$

If $\mathrm{Sk}_1\, X$ consists of edges a and b, then we attach a 2-cell along the loop following a, b, $-a$, and $-b$.

This CW structure comes from the quotient of a square picture in Example 1.2.8. In particular, in this figure



we see that

- all four vertices are identified to the same vertex p after the quotient, so there is only one 0-cell;
- the four edges of a square identify to two, so there is two 1-cell, labeled a and b; and
- there is one 2-cell, corresponding to the interior of the square.

Exercise 1.2.17. Realize the Klein bottle as a CW complex.

§1.2.4 Real and Complex Projective Spaces

In this subsection, we give a CW complex construction of the real and complex projective spaces.

▶ Real Projective Space

Definition 1.2.18 (Real projective space)**.**

The **real projective space** $\mathbb{RP}^n$ is the quotient

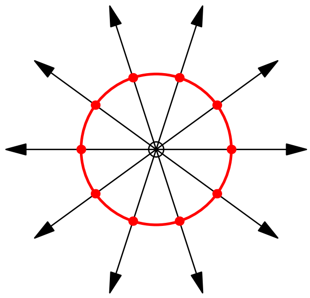
$$\frac{\mathbb{R}^{n+1} \setminus \{0\}}{a \sim \lambda a \text{ for all } a \in \mathbb{R}^{n+1} \setminus \{0\}, \text{ and } \lambda \in \mathbb{R} \setminus \{0\}}.$$

We can simplify this quotient in two ways.

- First, each point in a ray from 0 has a unique representative in S^n. Thus, we have that

$$\mathbb{RP}^n = \frac{S^n}{a \sim -a \text{ for all } a \in S^n}.$$

The following diagram depicts this process when $n = 1$. First, we reduce to points on the red circles. Then we identify opposite points together.



- Second, by *squishing* each hemisphere into a disc, we can identify opposite points in the disc to be together, giving only a single disc. Thus, $\mathbb{RP}^n$ is the same as a disk D^n where we identity opposite points on the boundary together.

However, note that the boundary of D^n is S^{n-1}, so the boundary of $\mathbb{RP}^n$ is S^{n-1} but we identify two opposite points together, which is precisely $\mathbb{RP}^{n-1}$! Hence, $\mathbb{RP}^n$ can be thought as attaching D^n onto $\mathbb{RP}^{n-1}$. The attaching map thus comes from the quotient projection $f : S^n \to \mathbb{RP}^{n-1}$.

This gives the CW structure of $\mathbb{RP}^n$ by induction, where we have $\mathrm{Sk}_{n-1}\,\mathbb{RP}^n = \mathbb{RP}^{n-1}$. There is one cell for each dimension from $0, 1, 2, \ldots, n$.

✍ **Exercise 1.2.19.**

(a) Explain why $\mathbb{RP}^1 \simeq S^1$.

(b) Realize $\mathbb{RP}^2$ is a quotient of a square similar to Example 1.2.8.

▶ **Complex Projective Space** We can define complex projective space similarly to the real projective space.

📋 **Definition 1.2.20** (Complex Projective Space)**.**

The **complex projective space** $\mathbb{CP}^n$ is the quotient

$$\frac{\mathbb{C}^{n+1} \setminus \{0\}}{a \sim \lambda a \text{ for all } a \in \mathbb{C}^{n+1} \setminus \{0\}, \text{ and } \lambda \in \mathbb{C} \setminus \{0\}}.$$

To give the cell structure to $\mathbb{CP}^n$, we go by a slighly different route. For each point $a = (a_0, a_1, \ldots, a_n) \in \mathbb{C}^{n+1}$, we split into two cases.

- If $a_0 \neq 0$, then each such point has a unique representation $(1 : a_1/a_0 : \cdots : a_n/a_0)$, so this part of the space is isomorphic to $\mathbb{C}^n \simeq \mathbb{R}^{2n}$, which is the same as interior of D^{2n}.

- If $a_0 = 0$, then this part of the space is given by $\mathbb{CP}^{n-1}$ by restrict to the remaining n coordinates.

Thus, $\mathbb{CP}^n$ **is obtained by attaching a $2n$-cell to $\mathbb{CP}^{n-1}$**. To write down the attaching map explicitly, we use the identification

$$\begin{aligned} D^{2n} &= \{(z_1, \ldots, z_n) \in \mathbb{C}^n : |z_1|^2 + \cdots + |z_n|^2 \leq 1\} \\ S^{2n-1} = \partial D^{2n} &= \{(z_1, \ldots, z_n) \in \mathbb{C}^n : |z_1|^2 + \cdots + |z_n|^2 = 1\}, \end{aligned}$$

and we let the points $(z_1, \dots, z_n)$ on the boundary be sent to point $(0 : z_1 : \cdots : z_n)$. Moreover, the bijection from $D^{2n} \setminus \partial D^{2n}$ to $\mathbb{CP}^n \setminus \mathbb{CP}^{n-1}$ is given by

$$\begin{aligned} f : D^{2n} \setminus \partial D^{2n} &\to \mathbb{CP}^n \setminus \mathbb{CP}^{n-1} \\ (z_1, \dots, z_n) &\mapsto (\sqrt{1 - |z_1|^2 - \cdots - |z_n|^2} : z_1 : \cdots : z_n) \\ g : \mathbb{CP}^n \setminus \mathbb{CP}^{n-1} &\to D^{2n} \setminus \partial D^{2n} \\ (w_0 : w_1 : \cdots : w_n) &\mapsto \left(\frac{w_1}{kw_0}, \dots, \frac{w_n}{kw_0} \right), \end{aligned}$$

where k is a positive real number such that

$$\frac{1}{k^2} = 1 - \frac{|w_1|^2 + \cdots + |w_n|^2}{k^2 |w_0|^2} \iff k = \sqrt{\frac{|w_0|^2 + |w_1|^2 + \cdots + |w_n|^2}{|w_0|^2}}.$$

By induction, we have obtained the cell structure of $\mathbb{CP}^n$ consisting of $n+1$ cells of dimensions $0, 2, 4, \dots, 2n$.

§1.3 Properties of Topological Spaces

§1.3.1 Hausdorff

In topology, every topological space you will care about is Hausdorff. It is a condition to make sure that every point is well-separated.

Definition 1.3.1.

A topological space X is **Hausdorff** if for every points $p \neq q$ in X, there exists open sets that separate p and q. In other words, there exists open sets $U, V \subseteq X$ such that $p \in U$, $q \in V$, and $U \cap V = \emptyset$.

Example 1.3.2.

Every subspace of $\mathbb{R}^n$ is Hausdorff. To see this, take any points $p \neq q \in \mathbb{R}^n$, and let $\varepsilon = 0.1|p - q|$. Then the balls $B_\varepsilon(p)$ and $B_\varepsilon(q)$ separates p and q.

Exercise 1.3.3. Prove that every subspace of Hausdorff space is Hausdorff.

We note without proof that every CW complex is Hausdorff. However, quotient of Hausdorff space need not be Hausdorff. An example is a **bug-eyed line**: we take disjoint union of two intervals $[0, 1]$, say that it is $[-1, 1] \times \{a, b\}$. Then we idenify (x, a) and (x, b) together for all $x \neq 0$. The resulting quotient will look like a single line, except that there are two copies of point 0 that share the neighborhoods in a weird fashion.

§1.3.2 Connectedness

Connectedness is a notion for a topological space to *be in a single piece.*

Definition 1.3.4.

A topological space X is **connected** if and only if the only sets that are both closed and open is X itself.

Equivalently, a topological space X is connected if and only if it cannot be written as $X = U \sqcup V$ where U and V are disjoint open sets.

Example 1.3.5.

In $\mathbb{R}$, union of intervals $(0,1) \cup [2,3]$ is not connected because $(0,1)$ and $[2,3]$ are both closed and open (in the induced topology).

It's not hard to see that a connected set $X \subseteq \mathbb{R}$ must be a single interval: if $a, b \in X$ and $a < x < b$, then x must be in X. This is because if $x \notin X$, then $X \cap (-\infty, x) = X \cap (-\infty, x]$ is both closed and open in X, which is a contradiction.

It's harder to show that a single interval is connected, which we do in the next proposition.

Proposition 1.3.6.

(a) The closed interval $[a, b]$ is connected.

(b) The open interval (a, b) is connected.

▶ *Proof.* This is the first nontrivial proof of this chapter.

(a) Assume for contradiction that $[a, b] = U \sqcup V$ for some open sets U and V. Without loss of generality, let $a \in U$. Define

$$T = \{u \in [a, b] : [a, u] \subseteq U\}, \quad m = \sup T.$$

We have two cases.

- If $m \in U$, then since U is open, there exists $\varepsilon > 0$ such that $(m - \varepsilon, m + \varepsilon) \subseteq U$. Then $m + \frac{\varepsilon}{2} \in T$, which is a contradiction to m being a supremum.
- If $m \in V$, then since V is open, there exists $\varepsilon > 0$ such that $(m - \varepsilon, m + \varepsilon) \subseteq V$. Thus, $T \cap (m - \varepsilon, m] = \emptyset$, implying that $m - \varepsilon$ is a smaller upper bound of T, contradicting that $m = \sup T$.

Both cases lead to a contradiction, so we are done.

(b) Assume for contradiction that $(a, b) = U \sqcup V$ for some open sets U and V. Take any decreasing sequence $x_1, x_2, \cdots \to a$ and increasing sequence $y_1, y_2, \cdots \to b$. Then $[x_i, y_i] = (U \cap [x_i, y_i]) \cup (V \cap [x_i, y_i])$, so by connectedness of $[x_i, y_i]$, we have that either $[x_i, y_i] \subseteq U$ or $[x_i, y_i] \subseteq V$. By restricting to an infinite sequence, we assume that $[x_i, y_i] \subseteq U$ for all i. Since $x_i \to a$ and $y_i \to b$, we get that $(a, b) \subseteq U$, which is a contradiction. ☐

By adapting a similar argument, it is straightforward all types of intervals $(-\infty, a)$, $(-\infty, a]$, (a, ∞), $[a, \infty)$, $(a, b]$, and $[a, b)$ are connected. Hence, connected subset of $\mathbb{R}$ is precisely the intervals.

The next exercise notes how connectedness interact with continuous maps.

Exercise 1.3.7.

(a) Suppose that $f : X \to Y$ is continuous and X is connected, prove that the image $f(X)$ must be connected as well.

(b) (Intermediate Value Theorem) Suppose that $f : [a, b] \to \mathbb{R}$ is a continuous function such that $f(a) < 0$ and $f(b) > 0$. Then there exists $x \in [a, b]$ such that $f(x) = 0$.

▶ **Path-connectedness** As one can see from an example of intervals, checking that a space is connected can be difficult. Fortunately, there is an easier criterion that suffices to guarantee that a space is connected.

Definition 1.3.8.

A topological space X is **path-connected** if and only if for any two points p, q, there exists a path from p to q, i.e., a continuous map $\gamma : [0,1] \to X$ such that $\gamma(0) = p$ and $\gamma(1) = q$.

Proposition 1.3.9.

Every path-connected topological space is connected.

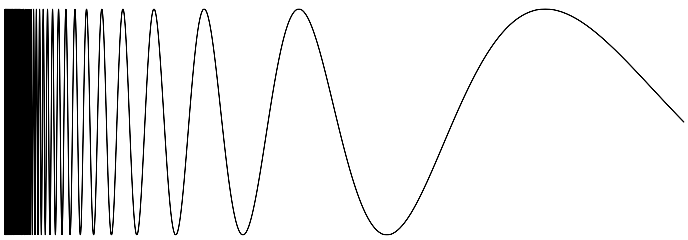
▶ *Proof.* Let X be a path-connected topological space and assume that X is not connected. Then $X = U \sqcup V$ for some open sets U and V. Take any point $p \in U$ and $q \in V$, and let $\gamma : [0,1] \to X$ be a path from p to q. Then $\gamma^{-1}U$ and $\gamma^{-1}V$ are disjoint open sets of $[0,1]$. We have that $0 \in \gamma^{-1}U$ and $1 \in \gamma^{-1}V$, so neither can be $[0,1]$. This contradicts the fact that $[0,1]$ is connected (Proposition 1.3.6). ☐

Example 1.3.10.

Every convex subset $X \subseteq \mathbb{R}^n$ is path-connected, hence connected. This is because any two points $p, q \in X$, the straight line through p and q is contained in X. (More formally, the path is $\gamma(t) = (1-t)p + tq$.)

Warning 1.3.11.

The converse of Proposition 1.3.9 is not true: every connected spaces need not be path-connected. A counterexample is the **topologist's sine curve**: which is the union of the origin $(0,0)$ and the graph $\left\{y = \sin(1/x) : x \in \left(0, \frac{1}{2\pi}\right]\right\}$.



It is connected because the graph $y = \sin(1/x)$ is connected, and one can show that adding point $(0,0)$ does not affect connectivity (similar to the argument that $[a,b]$ is connected above). It is not path-connected because point $(0,0)$ cannot reach anywhere.

We note one terminology that will occessionally appear in the future. Given any topological space X, the relation $\sim$ on X defined by

$$p \sim q \iff \text{there is a path from } p \text{ to } q$$

is an equivalence relation. (To show transitivity, we join the path from p to q and q to r to get a path from p to r. Can you write this down explicitly?) Thus, it splits X into equivalence classes, each is called a **path component** of X.

§1.3.3 Compactness

Compactness is a notion of boundedness. For example, $[0,1]$ is compact because it is bounded.

However, one needs to be careful since some spaces might look bounded when written in one way but not the other. For example, in Example 1.1.13, we have seen that the open interval $(0,1)$ is homeomorphic to $\mathbb{R}$. The former looks like it is bounded, but the latter is not. Since we want definition that is intrinsic to the topology, we would want to declare that $(0,1)$ is not bounded (i.e., not compact).

Here is the definition. It is not the most intuitive, but we will eventually work out some concrete examples.

Definition 1.3.12.

A topological space X is **compact** if and only if any open cover has a finite subcover. In other words, for any collection of open set $(U_i)_{i\in I}$ such that $\bigcup_{i\in I} U_i = X$, there exists a **finite** subset $J \subset I$ such that $\bigcup_{j\in J} U_j = X$.

Example 1.3.13.

$\mathbb{R}$ is not compact since the collection

$$\dots, (-1,\infty), (0,\infty), (1,\infty), (2,\infty), \dots$$

forms an open cover to $\mathbb{R}$, but there is no finite subset of those that cover $\mathbb{R}$.

Since $(0,1)$ is homeomorphic to $\mathbb{R}$, $(0,1)$ is not compact. One example of open cover of $(0,1)$ that does not have finite cover is

$$\left(\tfrac{1}{2},1\right), \left(\tfrac{1}{3},1\right), \left(\tfrac{1}{4},1\right), \dots.$$

This example also shows that $(0,1]$ is not compact. In particular, every interval is not compact unless it is closed.

On the other hand, $[0,1]$ is compact. As you might have guessed, showing this is much harder, and we will do it in a bit in Proposition 1.3.16.

▶ **Basic Properties.** We now begin proving properties of compact sets.

Proposition 1.3.14.

(a) If X is compact and $A \subseteq X$ is closed, then A is compact.

(b) Conversely, if X is Hausdorff and $A \subseteq X$ is compact, then A is closed.

(c) If X and Y is compact, then $X \times Y$ is compact. (Hence by induction, finite product of compact sets is compact.)

▶ *Proof.* (a) Definition-chasing. Given any open cover $(U_i)_{i\in I}$ of A, we form an open cover of X by including the open set $X \setminus A$. Now, we can pick a finite subcover, and it will naturally induces a finite subcover of A by removing $X \setminus A$.

(b) This is a little harder. We show that A is closed by writing it as an intersection of closed sets. To achieve this, for each point $p \in X \setminus A$, we have to get a closed set that contains A but does not contain p. Equivalently, (by taking complement), we want an open set that contains p but does not intersect A.

We do this using Hausdorff condition. For any point $q \in A$, we let U_q and V_q be such that $p \in U_q$, $q \in V_q$, but $U_q \cap V_q = \emptyset$. Then $(V_q)_{q\in A}$ forms an open cover of A, so by compactness of A, there exists finitely many points $q_1, \dots, q_n \in A$ such that $V_{q_1} \cup \dots \cup V_{q_n} = A$. We let $Y_p = \bigcap_{i=1}^n U_{q_i}$. We note that

- since $p \in U_{q_i}$ for all i, we have that $p \in Y_p$.
- since U_{q_i} does not intersect V_{q_i}, we have that Y_p does not intersect V_{q_i}, which implies that Y_p does not intersect A.

Now, we are almost done. We claim that

$$X \setminus A = \bigcup_{p\in X\setminus A} Y_p.$$

This follows from that $Y_p \subseteq X \setminus A$ for all p and we have that $p \in Y_p$ (hence in the union) for all $p \in X \setminus A$. Since Y_p is open, $X \setminus A$ is a union of open sets, hence open, so A is closed.

(c) Consider an open cover of $X \times Y$. By expressing each open set as a union of sets in form $U_i \times V_i$, we assume that our open cover is of the form $(U_i \times V_i)_{i \in I}$.

The strategy is as follows: for each point $p \in X$, we find a finite subcover of a small *tube* around p (of the form neighborhood around p times Y), and then we pick finite subset of those tubes to get the actual finite subcover.

Fix a point $p \in X$. Let $I_p = \{i \in I : p \in U_i\}$. Then $(V_i)_{i \in I_p}$ must be an open cover of Y, so we pick a finite subcover $V_{i_1}, \ldots, V_{i_n}$. We let $W_p = U_{i_1} \cap \cdots \cap U_{i_n}$. Then we have that $\bigcup_{j=1}^{n} U_{i_j} \times V_{i_j}$ contain $W_p \times Y$, so we have a finite subcover of $W_p \times Y$, giving our tube.

Note that $(W_p)_{p \in X}$ is an open cover of X, so we pick a finite subcover $W_{p_1}, \ldots, W_{p_m}$. Since each of $W_{p_1} \times Y, \ldots, W_{p_m} \times Y$ together cover X and has a finite subcover, it follows that we can take those m finite subcovers together to cover X, and we are done. □

Exercise 1.3.15. Let X be a topological space. Prove that any finite union of compact subsets of X is compact.

▶ **Compact subspaces of $\mathbb{R}^n$.** We now show that an interval is compact.

Proposition 1.3.16.

Any closed interval $[a, b]$ is compact. (Consequently, $[a, b]^n$ is compact by Proposition 1.3.14 (c).)

▶ *Proof.* We do the following nice trick. Fix an open cover $(U_i)_{i \in I}$. Consider the set

$$T = \{x \in [a, b] : [a, x] \text{ admits a finite subcover}\}.$$

First, note that $a \in T$ by taking any open set containing a. We will show that T is open, and that T is closed. This will allow us to conclude that $T = [a, b]$ by connectedness of $[a, b]$ (Proposition 1.3.6).

***T* is open.** Consider any $p \in T$ and take a finite subcover of $[a, p]$. In this finite subcover, there exists one open set U_i that contains p. In particular, there exists $\varepsilon > 0$ such that $(p - \varepsilon, p + \varepsilon)$ is contained in U_i. Thus, this subcover is automatically a finite subcover of $[a, p + \varepsilon/2]$, implying that $(p - \varepsilon/2, p + \varepsilon/2) \subseteq T$, showing that T is open.

***T* is closed.** Now, we consider $p \notin T$ and take U_i such that $p \in U_i$. There exists $\varepsilon > 0$ such that $(p - \varepsilon, p + \varepsilon) \subseteq U_i$. We claim that T does not intersect $(p - \varepsilon/2, p + \varepsilon/2)$. To see this, assume that there exists $q \in (p - \varepsilon/2, p + \varepsilon/2)$ such that $q \in T$. Then $[a, q] \cup U_i$ contains $[a, q] \cup (p - \varepsilon, p + \varepsilon)$, which contains $[a, p]$. Hence, $p \in T$, which is a contradiction. This shows that T is closed. □

Corollary 1.3.17 (Heine-Borel Theorem)**.**

A subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded. (Bounded means contained in $[a, b]^n$ for some $a, b \in \mathbb{R}$.)

▶ *Proof.* ($\Leftarrow$) Since $[a, b]^n$ is compact, by Proposition 1.3.16, any closed subset of $[a, b]^n$ is compact by Proposition 1.3.14 (a).

($\Rightarrow$) Closed follows from Proposition 1.3.14 (b). Now we only need to show that any compact set X is bounded. To do this, consider an open cover

$$(-1, 1)^n, (-2, 2)^n, (-3, 3)^n, \ldots.$$

There must exist a finite subcover, which means that $(-m, m)^n$ must contain X. This means $X \subseteq [-m, m]^n$, so X is bounded. □

▶ **Compactness and Continuous Map** The next exercise shows how compactness behaves under continuous maps.

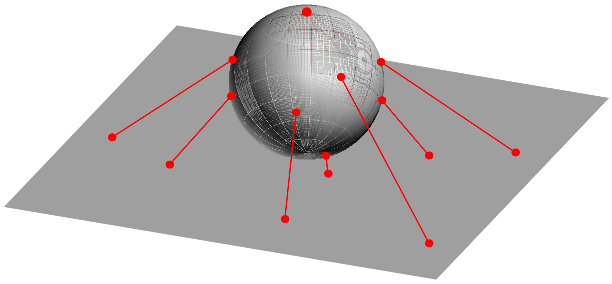
✍ **Exercise 1.3.18.**

(a) Let $f : X \to Y$ be a continuous function where X is compact. Then the image $f(X)$ is compact.

(b) (Extreme value theorem) Let X be a compact topological space and $f : X \to \mathbb{R}$ be continuous. Then f is bounded and has a maximum / minimum value.

§1.4 Problems

Problem 1.A (Stereographic Projection)**.** Prove that $S^n \setminus \{\text{point}\}$ and $\mathbb{R}^n$ are homeomorphic by writing out explicit homeomorphism. The following is an example of homeomorphism when $n = 2$.



Problem 1.B (Connected Sum)**.** A topological space M is an n-**manifold** if it is Hausdorff and for every $x \in M$, there exists an open set U containing x such that $U \simeq \mathbb{R}^n$. Example of 2-manifolds are Σ_g (g-holed torus), the Klein bottle, and the real projective plane.

Given two n-manifolds X and Y, the **connected sum** $X \# Y$ is defined by removing a small open neighborhood $U \subseteq X$ and $V \subseteq Y$ and identify the boundaries ∂U and ∂V (both isomorphic to S^{n-1}) together.

(a) Convince yourself that $\Sigma_g \# \Sigma_h \simeq \Sigma_{g+h}$.

(b) Prove that $\mathbb{RP}^2 \# \mathbb{RP}^2$ is homeomorphic to the Klein bottle.

Problem 1.C (Pasting Lemma)**.** Let X and Y be topological spaces, and let $(U_i)_{i \in I}$ be a collection of open sets of X. Suppose that there is a collection of continuous maps $f_i : U_i \to Y$ such that for all $i, j \in I$, the restrictions $f_i|_{U_i \cap U_j}$ and $f_j|_{U_i \cap U_j}$ are equal. Prove that there exists a map $f : X \to Y$ such that $f|_{U_i} = f_i$ for all i.

Problem 1.D. Let X be a topological space. Prove that X is Hausdorff if and only if the diagonal

$$\Delta = \{(x, x) : x \in X\} \subseteq X \times X$$

is a closed subset of $X \times X$.

Problem 1.E. A topological space X is **locally path-connected** if for any point x and open set $U \ni x$, there exists a path-connected open set V such that $x \in V \subseteq U$.

Prove that any topological space X that is connected and locally path-connected must be path-connected.

Problem 1.F (Lebesgue Covering Lemma)**.** Let X be a compact subset of $\mathbb{R}^n$, and let $\mathcal{U}$ be an open cover of X. Prove that there exists a real number $\varepsilon > 0$ such that for any point $x \in X$, the ball $B_\varepsilon(x)$ is contained in U for some $U \in \mathcal{U}$.

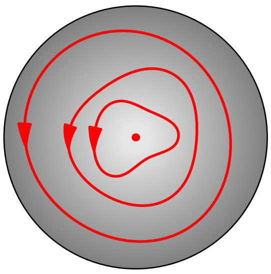
Problem 1.G. Prove that any CW complex with finitely many cells is compact.

2 Fundamental Groups

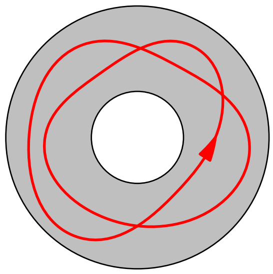
The major question that motivates algebraic topology is **are two topological spaces homeomophic (or homotopy equivalent)?** If the answer is yes, all we need is to write down the map. However, we would like to be able to prove that there is no such map as well.

A standard way to do this is to **define algebraic invariants** of a topological spaces that is invariant under homeomorphism. If we know that these invariants are different, then we know that the corresponding topological spaces are not homeomorphic.

The algebraic invariant that we will meet this chapter is the **fundamental group**. Informally, given a topological space X, its fundamental group, denoted $\pi_1(X)$, is a group of loop up to deformation, and the group operation is concatenating two loops. For example, in a sphere S^2, every loop can be deformed to the trivial loop (i.e., a point), so $\pi_1(S^2) = 0$.



Deforming loops in S^2 (top view)



A loop in the annulus A that cannot be deformed.

On the other hand, the fundamental group of an annulus A (see the picture above) is $\mathbb{Z}$. This is because for each $n \in \mathbb{Z}$, the loop that winds around the center n times (negative number of times means winding in the opposite direction) cannot be deformed to each other. Thus, the fundamental group of A and S^2 are different, so A and S^2 are not isomorphic.

In order to formalize this, we first precisely define the phrase "up to deformation". This is done in Section 2.1. Then in the following Section 2.2, we define the fundamental group and prove some basic property. Finally, in Section 2.3, we state and prove our central result: the Seifert-van Kampen theorem, which allows us to compute various fundamental groups by breaking the space down into smaller spaces.

§2.1 Homotopy

§2.1.1 Homotopic Maps

Definition 2.1.1 (Homotopic map).

Let X and Y be topological spaces. Two continuous maps $f, g : X \to Y$ in **Top** are **homotopic** if there exists

$$H : X \times [0, 1] \to Y$$

such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$. The map H is called the **homotopy**.

Informally, the map H represents a continuous deformation, and the second input (the one in $[0,1]$) represents the *time* of the continuous deformation.

Proposition 2.1.2 (Homotopy is an equivalence relation).

Let X and Y be topological spaces. Suppose that for any two maps $f,g : X \to Y$, we have $f \sim g$ if and only if f and g are homotopic. Then $\sim$ is an equivalence relation.

▶ *Proof.* We have to check three things.

- **Reflexivity.** f is homotopic to itself because of the homotopy $H(x,t) = f(x)$.
- **Symmetry.** if $f \sim g$ with homotopy H, then $g \sim f$ with homotopy $H'(x,t) = H(x,1-t)$, which has $H'(x,0) = H(x,1) = g(x)$ and $H'(x,1) = H(x,0) = f(x)$.
- **Transitivity.** suppose $H_1 : f \simeq g$ and $H_2 : g \simeq h$ are homotopies from f to g and from g to H. Then,
$$H(x,t) = \begin{cases} H_1(x,2t) & 0 \le t \le 1/2 \\ H_2(x,2t-1) & 1/2 \le t \le 1 \end{cases}$$
is a homotopy from f to h. □

§2.1.2 Homotopy Equivalence

Homotopy equivalence is a more robust notion of homeomorphism. Homeomorphism is a continuous deformation, but homotopy equivalence allows more robust actions such as squeezing a line to a point.

To define this, we use the idea of homotopic maps. When defining homeomorphism, we require two maps $f : X \to Y$ and $g : Y \to X$ such taht $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$. Being homotopy equivalence relaxes this condition to only that $g \circ f$ and $f \circ g$ are homotopic to the identity maps. This leads to the following definition.

Definition 2.1.3 (Homotopy equivalent spaces).

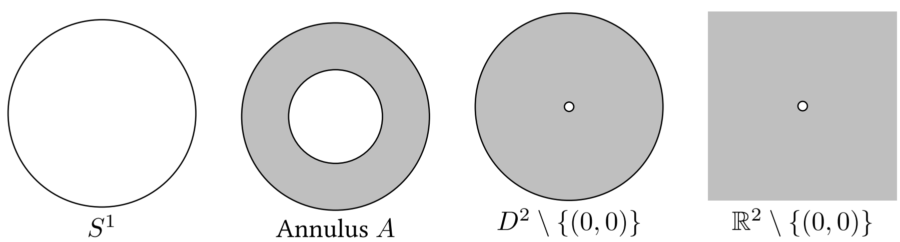
Two topological spaces X and Y are **homotopy equivalent** if there are continuous maps $f : X \to Y$ and $g : Y \to X$ such that

- $g \circ f$ is homotopic to id_X
- $f \circ g$ is homotopic to id_Y.

The maps f and g are called **homotopy equivalence**.

Example 2.1.4.

The following four spaces are homotopy equivalent.



Let us first show that S^1 is homotopy equivalent to A. To do this, take the coordinate system such that

$$S^1 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$$
$$A = \{(x,y) \in \mathbb{R}^2 : 0.5 \leq x^2 + y^2 \leq 1.5\}.$$

Then, the homotopy equivalence is as follows:

$$\begin{aligned} f : S^1 &\to A \\ (x,y) &\mapsto (x,y) \end{aligned} \quad \text{and} \quad \begin{aligned} g : A &\to S^1 \\ (x,y) &\mapsto \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}\right). \end{aligned}$$

Obviously, $g \circ f = \mathrm{id}_{S^1}$. Thus, we have to show that the map $f \circ g$, which sends (x,y) to $\left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}\right)$ is homotopic to the identity map. Geometrically, this map sends every point in the annulus to the sphere S^1. Thus, our homotopy will look like we keep shrinking A to make it thinner. Motivated by this, we write the homotopy

$$H((x,y),t) = \left((1-t) + \frac{t}{r}\right)(x,y),$$

where $r = \sqrt{x^2+y^2}$. It is easy to see that this works.

We leave proving that the second, third, and fourth spaces are homotopy equivalent as an exercise. As a hint, the homotopy equivalence from A to $D^2 \setminus \{0,0\}$ is by shrinking the inner hole of the annulus. The homotopy equivalence from $D^2 \setminus \{(0,0)\}$ to $\mathbb{R}^2$ is by expanding the disk. See if you can translate those into explicit equations.

By far the most common type of homotopy equivalence (applies to all the previous examples) is deformation retract.

Definition 2.1.5 (Deformation retract).

A subspace $A \subseteq X$ is a **deformation retract** of X if there exists a continuous map $H : X \times [0,1] \to X$ such that

- $H(x,0) = x$ for all $x \in X$;
- $H(x,1) \in A$ for all $x \in X$;
- $H(a,t) = a$ for all $a \in A$ and $t \in [0,1]$.

In this case, we can see that if $r(x) = H(x,1)$, then is a map from X to A. Moreover, if i is the inclusion from A to X, then $r \circ i = \mathrm{id}_A$ and $i \circ r$ is homotopic to id_X by the homotopy h. Thus, **A is a deformation retract of X implies that A is homotopy equivalent to X**.

▶ **The Homotopy Category.** We give a categorical interpretation of homotopy equivalence.

Definition 2.1.6 (Homotopy Category).

The **homotopy category** **HoTop** is the category with

- **objects:** topological spaces
- **morphisms:** for any topological spaces X and Y, the morphisms are the equivalence classes of continuous map $f : X \to Y$, where two maps are equivalent if they are homotopic.

In particular, we see that two spaces are homotopy equivalent if and only if they are isomorphic in **HoTop**.

§2.2 Fundamental Groups

§2.2.1 Definition

Definition 2.2.1 (Fundamental Group)**.**

Given a topological space X and a basepoint $x_0 \in X$, the **fundamental group** $\pi_1(X, x_0)$ is the group

$$\pi_1(X, x_0) = \frac{\{\text{continuous map } f : [0,1] \to X \text{ such that } f(0) = f(1) = x_0\}}{\sim},$$

where $f \sim g$ if and only if f and g are homotopic as loops. In other words, there exists a homotopy $H : [0,1]^2 \to X$ such that $H(t,0) = f(t)$, $H(t,1) = g(t)$, and $H(0,s) = H(1,s) = x_0$.

A continuous map $f : [0,1] \to X$ with $f(0) = f(1) = x_0$ is called a **loop** at x_0. The image of a loop f in $\pi_1(X, x_0)$ is denoted $[f]$.

The group operation is defined as follows: given two loops f, g, we define a loop $f \cdot g$ by concatenating f and g:

$$(f \cdot g)(t) = \begin{cases} f(2t) & \text{if } 0 \le t \le \frac{1}{2} \\ g(2t-1) & \text{if } \frac{1}{2} \le t \le 1. \end{cases},$$

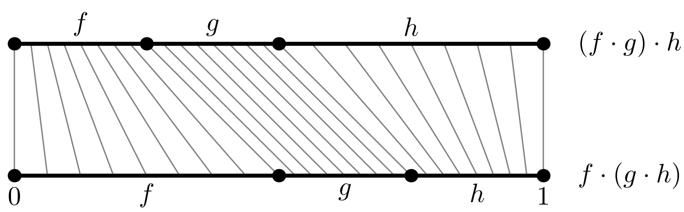
and the group operation is $[f][g] = [f \cdot g]$. We omit the proof that $f \cdot g$ is continuous.

Warning 2.2.2.

The condition that $[f]$ and $[g]$ are the same in $\pi_1(X, x_0)$ is not the same as f and g are homotopic. In particular, we a homotopy between f and g, and **that homotopy must remain a loop at x_0 thoughout**.

We first show that it is a group. This verification is rather tedious.

- **Associativity.** Given any loops f, g, h, we have to show that $[(f \cdot g) \cdot h] = [f \cdot (g \cdot h)]$. Unfortunately, loops $(f \cdot g) \cdot h$ and $f \cdot (g \cdot h)$ are not the same. In particular, in $(f \cdot g) \cdot h$ the f part occupies $[0, 1/4]$, the g part occupies $[1/4, 1/2]$, and the h part occupies $[1/2, 1]$. In contrast, in $f \cdot (g \cdot h)$, the f part occupies $[0, 1/2]$, the g part occupies $[1/2, 3/4]$, and the h part occupies $[3/4, 1]$.



  The saving grace is that we are modding out by homotopy. In particular, it is easy to write a homotopy equivalence from $(f \cdot g) \cdot h$ to $f \cdot (g \cdot h)$ by

  - gradually shift the boundary between f and g from $\frac{1}{4}$ to $\frac{1}{2}$, and
  - gradually shift the bounday between g and h from $\frac{1}{2}$ to $\frac{3}{4}$.

Upon realizing this, we can write out explicit equation as

$$H(t,s)=\begin{cases} f\left(\frac{4}{s+1}\cdot t\right) & \text{if } 0\le t\le \frac{s+1}{4} \\ g\left(4\left(t-\frac{s+1}{4}\right)\right) & \text{if } \frac{s+1}{4}\le t\le \frac{s+2}{4} \\ h\left(\frac{t-(s+2)/4}{1-(s+2)/4}\right) & \text{if } \frac{s+2}{4}\le t\le 1 \end{cases}.$$

This is slightly messy, but the more important thing is that it works.

- **Identity.** The identity is the trival loop

$$\begin{aligned} e : [0,1] &\to X \\ t &\mapsto x_0. \end{aligned}$$

For any loop f, the loop $e\cdot f$ stays at x_0 from time 0 to $\frac{1}{2}$. To get the homotopy from $e\cdot f$ to f, we just need to shrink the time that the loop stays. Writing out explicit equation, the following is a homotopy from f to $e\cdot f$:

$$H(t,s)=\begin{cases} x_0 & \text{if } 0\le t\le \frac{s}{2} \\ f\left(\frac{t-s/2}{1-s/2}\right) & \text{if } \frac{s}{2}\le t\le 1. \end{cases}.$$

Thus, $[f]=[e][f]$. Analogous argument shows that $[f][e]=[f]$, completing the proof of identity.

- **Inverse.** The inverse of loop f is obtained by reversing f to get

$$\overline{f}(t)=f(1-t).$$

To get homotopy equivalence from $f\cdot\overline{f}$ to e, we just need to *cancel out* the part that are reverse to each other. In particular, we have that

$$H(t,s)=\begin{cases} f(2st) & 0\le t\le \frac{1}{2} \\ f(2s(1-t)) & \frac{1}{2}\le t\le 1 \end{cases}$$

is a homotopy from e to $f\cdot\overline{f}$, showing that $[f][\overline{f}]=[e]$. Analogously, one can show that $[\overline{f}][f]=[e]$, completing the proof that it is an inverse.

Example 2.2.3.

If X is a convex subset of $\mathbb{R}^n$, then $\pi_1(X,x_0)=1$ because every loop f has a homotopy equivalent to the trivial loop by

$$H(t,s)=sx_0+(1-s)f(t).$$

In particular, $\pi_1(\mathbb{R}^n,x_0)=1$ and $\pi_1(D^n,x_0)=1$.

Definition 2.2.4.

A loop f is **nulhomotopic** if and only if $[f]$ is the identity in $\pi_1(X,x_0)$.

§2.2.2 Basic Properties

We now prove various properties of the fundamental group.

▶ **Independence of Basepoint.** If X is path-connected, then it turns out that the choice of base point x_0 does not matter.

Proposition 2.2.5.

Let X be a topological space, and suppose that x_0 and x_1 are connected by a path. Then $\pi_1(X, x_0) \simeq \pi_1(X, x_1)$.

In particular, if X is path-connected, then $\pi_1(X, x_0)$ does not depend on x_0, and from now on, we will only write $\pi_1(X)$ with basepoint unspecified.

▶ *Proof.* Let $p : [0,1] \to X$ be a path from x_0 to x_1. Our bijection would take a loop f at x_0 to a loop at x_1 that

- follow the reverse of p to x_0;
- follow f along a loop; and
- follow p back to x_1.

In equations, this is

$$g(t) = \begin{cases} p(1-3t) & \text{if } 0 \le t \le \frac{1}{3} \\ f(3t-1) & \text{if } \frac{1}{3} \le t \le \frac{2}{3} \\ p(3t-2) & \text{if } \frac{2}{3} \le t \le 1. \end{cases}$$

We leave the reader to write out the inverse map and verify that this is indeed an isomorphism. □

We make the following definition.

Definition 2.2.6.

A topological space X is **simply connected** if it is path-connected and $\pi_1(X)$ is trivial.

▶ **The π_1 functor.** We promote π_1 to a functor.

Definition 2.2.7 (The Category of Pointed Spaces)**.**

The category $\mathbf{Top}_\bullet$ of **pointed spaces** is a category with

- **objects:** pair (X, x_0) where X is a topological space and $x_0 \in X$.
- **morphisms:** morphisms $f : (X, x_0) \to (Y, y_0)$ is a continuous map $f : X \to Y$ such that $f(x_0) = y_0$.

Proposition 2.2.8.

π_1 can be promoted to a functor from $\mathbf{Top}_\bullet \to \mathbf{Groups}$.

▶ *Proof.* In order to define a functor, we have to describe how to map morphisms. In particular, given a map $f : (X, x_0) \to (Y, y_0)$ in $\mathbf{Top}_\bullet$, we have to construct a morphism $\pi_1(f) : \pi_1(X, x_0) \to \pi_1(Y, y_0)$. To do this, suppose that γ is a loop in X at x_0, we let $\pi_1(f) = [f \circ \gamma]$. We have to check that this is well-defined. Suppose that γ and γ' are homotopic with homotopy H. Then $f \circ H$ is a homotopy from $f \circ \gamma$ to $f \circ \gamma'$, implying that $\pi_1(f)$ is well-defined.

It is easy to see that $\pi_1(f)$ satisfies all axioms for a functor. □

By the independence of basepoint, we also have the functor

$$\pi_1 : \{\text{path-connected topological spaces}\} \to \mathbf{Groups}.$$

▶ **Homotopy Invariance.**

Proposition 2.2.9 (Homotopy Invariance)**.**

Suppose that $f, g : (X, x_0) \to (Y, y_0)$ are maps of pointed spaces. If f and g are homotopic (this means that there exists $H : X \times [0,1] \to Y$ such that $H(x,0) = f(x)$, $H(x,1) = g(x)$, and $H(x_0, t) = y_0$), then $\pi_1(f) = \pi_1(g)$.

▶ *Proof.* This is fairly clear. Given any γ, we have to show that $f \circ \gamma$ and $g \circ \gamma$ are homotopic, but the homotopy that works is $H'(t,s) = H(\gamma(t), s)$, which satisfies $H'(t,0) = f(\gamma(t))$ and $H'(t,1) = g(\gamma(t))$. Hence, we are done. □

Corollary 2.2.10.

If (X, x_0) and (Y, y_0) are homotopy-equivalent (X and Y are homotopy equivalent and the homotopy equivalence sends x_0 to y_0), then $\pi_1(X, x_0) \simeq \pi_1(Y, y_0)$.

(In particular, if X and Y are path-connected and homotopy-equivalent, then $\pi_1(X) \simeq \pi_1(Y)$.)

▶ *Proof.* Suppose that $f : (X, x_0) \to (Y, y_0)$ and $g : (X, x_0) \to (Y, y_0)$ are homotopy equivalence. Then we have maps $\pi_1(f) : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ and $\pi_1(g) : \pi_1(Y, x_0) \to \pi_1(X, y_0)$. Since $f \circ g$ is homotopic to identity, we get by the previous proposition that $\pi_1(f \circ g) = \pi_1(f) \circ \pi_1(g)$ is the identity map. Similarly, $\pi_1(g) \circ \pi_1(f)$ is the identity map, so $\pi_1(f)$ and $\pi_1(g)$ are inverses. □

Exercise 2.2.11. Prove that $\pi_1(X \times Y) \simeq \pi_1(X) \times \pi_1(Y)$.

§2.2.3 Fundamental Group of Circle

In this section, we prove the first nontrivial fundamental group.

Theorem 2.2.12.

We have $\pi_1(S^1) = \mathbb{Z}$.

▶ *Proof.* View S^1 as the complex unit circle $\{z : |z| = 1\}$. The reason why we expect $\pi_1(S^1)$ to be $\mathbb{Z}$ is because for each $n \in \mathbb{Z}$, there is a loop that *winds* around S^1 exactly n times (if n is positive, then it winds counterclockwise, and if n is negative, then it winds clockwise). More precisely, we have a map

$$\begin{aligned} \gamma : \mathbb{Z} &\to \pi_1(S^1) \\ n &\mapsto \big(t \mapsto (\cos(2\pi n t), \sin(2\pi n t))\big). \end{aligned}$$

It is easy to see that this map is a group homomorphism by checking how loops $\gamma(m)$ and $\gamma(n)$ concatenate. The hard task is to check that this map is bijective. In particular, we must show that

- (injectivity) the loop $\gamma(n)$ is not nulhomotopic for all $n \neq 0$.
- (surjectivity) every loop f is homotopic to $\gamma(n)$ for some $n \in \mathbb{Z}$.

In order to show this, we use some facts about maps in S^1. We could have proved these now, but it will be a special case of Theorem 3.1.8, so the proof will be deferred to the general result there. We define the map

$$\begin{aligned} p : \mathbb{R} &\to S^1 \\ \theta &\mapsto (\cos(2\pi\theta), \sin(2\pi\theta)), \end{aligned}$$

which wraps the real line to a sphere. Later we will see that p has a special property that it is a **covering map**. We note the follows facts.

(a) For any map $f : [0,1] \to S^1$ and any $\widetilde{x}_0 = p^{-1}(x_0)$ (where $x_0 = f(0)$), there exists unique map $\widetilde{f} : [0,1] \to \mathbb{R}$ such that $\widetilde{f}(0) = \widetilde{x}_0$ and $p \circ \widetilde{f} = f$.

(Such map $\widetilde{f}$ is called a **lift** of f.)

(b) For any homotopy $H : [0,1]^2 \to S^1$ with $H(0,s) = x_0$ for all s and any $\widetilde{x}_0 = p^{-1}(x_0)$, there exists unique map $\widetilde{H} : [0,1]^2 \to \mathbb{R}$ such that $\widetilde{H}(0,s) = \widetilde{x}_0$ for all s and $p \circ \widetilde{H} = H$.

To prove surjectivity, we choose the base point to be $(1,0)$ and consider any loop $f : [0,1] \to S^1$ such that $f(0) = f(1) = (1,0)$. Taking $\widetilde{f}$ in (a) to be such that $p \circ \widetilde{f} = f$ and $\widetilde{f}(0) = 0$, we get that $\widetilde{f}(1) = n$ for some $n \in \mathbb{Z}$. Moreover, the map $\widetilde{f}$ is homotopic to a path with constant speed from 0 to n. Applying p, we find that f is homotopic to $\gamma(n)$.

To prove injectivity, we assume that there is a homotopy H from $\gamma(n)$ to the trivial loop at $(1,0)$. Take any lift $\widetilde{H}$ such that $\widetilde{H}(0,s) = 0$. Then $\widetilde{H}(t,0)$ is a lift of $\gamma(n)$, which implies that $\widetilde{H}(t,0) = nt$, so $\widetilde{H}(1,0) = n$. Similarly, we have $\widetilde{H}(t,1)$ is a lift of the trivial loop, so $\widetilde{H}(1,1) = 0$. However, $\widetilde{H}(1,t)$ must be on $p^{-1}(0) = \mathbb{Z}$, so since $\mathbb{Z}$ is discrete, it follows that $\widetilde{H}(1,t)$ is constant. This forces $n = 0$.

This completes the proof that $\pi_1(S^1) = \mathbb{Z}$, modulo the facts (a) and (b) above. □

As a corollary, we deduce (using Exercise 2.2.11) that the fundamental group of a torus is $\mathbb{Z}^2$.

§2.2.4 Application: Fundamental Theorem of Algebra

The fact that $\pi_1(S^1) = \mathbb{Z}$ can be use to give a short proof of the fundamental theorem of algebra. Arguably, this is the most intuitive proof of the theorem.

Theorem 2.2.13 (Fundamental theorem of algebra).

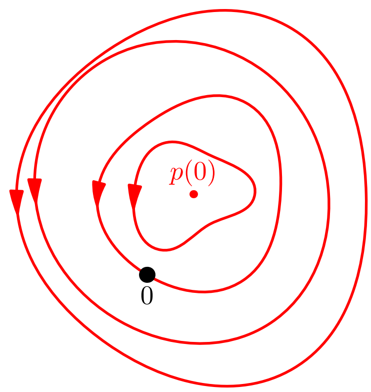
Let $p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial where $a_{n-1}, \ldots, a_1, a_0$ are complex numbers. Then there exists a complex number r such that $p(r) = 0$.

Before seeing the formal proof, let us explain the idea informally so that we can see how topology comes into play. We assume for a contradiction that $p(r) \neq 0$ for all complex number r. The key idea is to look at the loop γ_R defined by

$$t \mapsto p(Re^{2\pi it}) \quad \text{for } t \in [0,1],$$

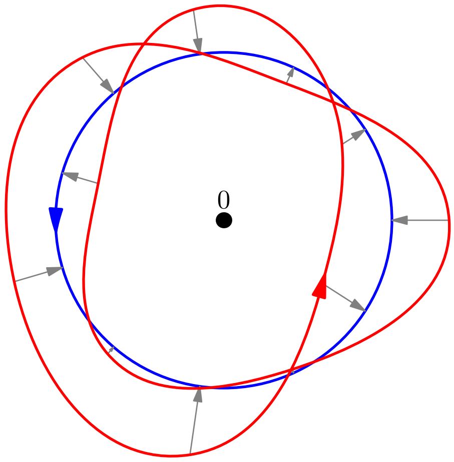
which is the image of the circle with radius R under p.

- As $R \to 0$, γ_R shrinks to a very small loop at $p(0) \neq 0$. In particular, γ_R should not wind around 0 because otherwise, it will hit 0 at some point as R shrink. This contradicts that $p(r) \neq 0$ for all r.



Any loop that winds around 0 must hit 0 while deforming to a point.

- On the other hand, if R is large, then $p(R\,e^{2\pi i t})$ is approximately the term with largest degree, $(R\,e^{2\pi i t})^n$, and we know that the loop $t \mapsto (R\,e^{2\pi i t})^n$ winds around the circle n times.



Deforming a loop of polynomial (red) to a loop $z \mapsto (R\,e^{i\pi t})^n$ (blue).

Both bullet points lead to a contradiction.

To formalize the "wind around 0", we can use the fundamental group. This results in the following proof.

▶ *Proof.* Assume for the contradiction that $p(r) \neq 0$ for all complex number r. Thus, p is a continuous map $p : \mathbb{C} \to \mathbb{C} \setminus \{0\}$. Since $\mathbb{C} \setminus \{0\} \simeq \mathbb{R}^2 \setminus \{0\}$ is homotopy equivalent to S^1 (see Example 2.1.4) we get by Theorem 2.2.12 that $\pi_1(\mathbb{C} \setminus \{0\}) = \mathbb{Z}$. Since $\pi_1(\mathbb{C}) = \pi_1(\mathbb{R}^2) = 0$, we get that the induced map $\pi_1(p) : \pi_1(\mathbb{C}) \to \pi_1(\mathbb{C} \setminus \{0\})$ is a zero map.

On the other hand, take $R > n \max(|a_{n-1}|, \ldots, |a_1|, |a_0|)$ and consider the loop

$$\begin{aligned} \gamma : [0,1] &\to \mathbb{C} \\ t &\mapsto R\,e^{2\pi i t} \end{aligned}$$

along the unit circle of radius R. We claim that $p \circ \gamma$ is homotopic to the loop that sends t to $(R\,e^{2\pi i t})^n$. The relevant homotopy is

$$(s,t) \mapsto s \cdot p(R\,e^{2\pi i t}) + (1-s) \cdot (R\,e^{2\pi i t})^n,$$

which never pass through 0 since the term $(R\,e^{2\pi i t})^n$ will dominate the rest if R is large enough. Thus, $p \circ \gamma$ corresponds to element n in $\pi_1(\mathbb{C} \setminus \{0\})$, which is a contradiction to the first paragraph. □

§2.3 The Seifert-van Kampen Theorem

§2.3.1 Coproduct and Pushout of Groups

Before we state the theorem, we discuss some group-theoretic constructions.

▶ Coproduct of Groups.

Definition 2.3.1.

Given a collection of groups $\{G_\alpha\}_{\alpha \in A}$, the **coproduct** $\coprod_{\alpha \in A} G_\alpha$ is a group, together with the canonical inclusion maps $\iota_\alpha : G_\alpha \to \coprod_{\alpha \in A} G_\alpha$ that satisfies the following universal property:

"for any group H and group homomorphisms $\phi_\alpha : G_\alpha \to H$ for each $\alpha \in A$, there exists unique group homomorphism $\phi : \coprod_{\alpha \in A} G_\alpha \to H$ that commutes with ι_α and ϕ_α for all α."

By the usual universal property argument, coproduct is unique up to isomorphism. We can construct this explicitly as follows: define $\coprod_{\alpha \in A} G_\alpha$ to be the set of strings $g_{\alpha_1} \cdots g_{\alpha_n}$ such that

- $\alpha_i \in A$ for all i.
- $\alpha_i \neq \alpha_{i+1}$ for all i.
- g_{α_i} is a non-identity element in G_{α_i}.

The group operation is by concatenating these two strings and collapse two elements from the same group into one. One can easily check that this satisfies the universal property.

In the case where $G_\alpha = \mathbb{Z}$ for all α, we get that $\coprod_{\alpha \in A} \mathbb{Z}$ is the **free group** $F(A)$ indexed by A. A special case of this is when $A = \{1, 2, \ldots, n\}$, which we denote the group by F_n, the free group with n elements.

The free product of finitely many groups $A_1, \ldots, A_n$ is often denoted $A_1 \amalg \cdots \amalg A_n$.

► **Pushout of Groups.**

Definition 2.3.2.

Given a diagram of groups and group homomorphisms

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"] & B \\ C & \end{tikzcd},$$

the **pushout** of this diagram is a group D with a map $B \to D$ and $C \to D$ such that the following square commutes

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd},$$

and it is universal, i.e., for any group G and homomorphism $B \to G$ and $C \to G$ that makes the diagram below commutes, there exists a unique group homomoprhism $D \to G$ that makes the diagram all commutes.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"] & B \arrow[d] \arrow[ddr, bend left] & \\ C \arrow[r] \arrow[drr, bend right] & D \arrow[dr, dashed, "\exists!"] & \\ & & G \end{tikzcd}.$$

We say that the square with A, B, C, and D form a **pushout square** if D is the pushout of A, B, and C.

The universal property above characterizes pushouts up to isomorphism. The explicit construction is the following:

$$D = \frac{B \amalg C}{\langle f(a) g(a)^{-1} : a \in A \rangle}.$$

(In other words, we start with free product $B \amalg C$ and mod out by relations $f(a) = g(a)$ for all $a \in A$.) One can check that this satisfies the universal property.

§2.3.2 Statement and Examples

We now state the theorem.

Theorem 2.3.3 (Seifert-van Kampen Theorem)**.**

Let X be a path-connected topological space such that $X = U \cup V$ where U and V are open and path-connected sets, and $U \cap V$ is path-connected. Then we have the pushout square

$$\begin{tikzcd} \pi_1(U \cap V) \arrow[r] \arrow[d] & \pi_1(U) \arrow[d] \\ \pi_1(V) \arrow[r] & \pi_1(X) \end{tikzcd},$$

where all maps are induced from the inclusions $U \cap V \to U$, $U \cap V \to V$, $U \to X$, and $V \to X$ by the π_1 functor.

In particular, if $i_1 : U \cap V \to U$ and $i_2 : U \cap V \to V$, we get that

$$\pi_1(X) = \frac{\pi_1(U) \amalg \pi_1(V)}{\langle \pi_1(i_1)(a) = \pi_1(i_2)(a) \text{ for all } a \in \pi_1(U_1 \cap U_2)\rangle}.$$

Intuitively, the theorem is saying that a loop in X is generated by loops in U and V, but we must be make sure that loops in $U \cap V$ get sent to the same thing in U and in V. The theorem tells you that there is nothing else you have to worry about.

Remark 2.3.4 (Seifert-van Kampen for more than two spaces)**.**

There is a version of Seifert-van Kampen where we have a union of more than two spaces. It states as follows:

Let X be a path-connected topological space and $x_0 \in X$. Let $X = \bigcup_{\alpha \in I} A_\alpha$ where A_i are open and $x_0 \in A_\alpha$. Suppose that A_α, $A_\alpha \cap A_\beta$, and $A_\alpha \cap A_\beta \cap A_\gamma$ are path-connected for all $\alpha, \beta, \gamma \in I$. Let $i_{\alpha\beta} : A_\alpha \cap A_\beta \to A_\alpha$ be the inclusion map. Then

$$\pi_1 X = \frac{\coprod_{\alpha \in I} \pi_1(A_\alpha)}{\langle \pi_1(i_{\alpha\beta})(a) = \pi_1(i_{\beta\alpha})(a) \text{ for all } a \in \pi_1(A_\alpha \cap A_\beta)\rangle}$$

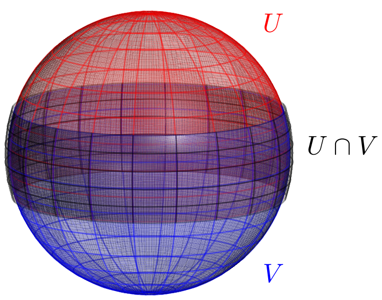
We will sketch the proof of this theorem in the next subsection. For now, we aim to provide example calculations.

The simplest case happens when $\pi_1(U) = \pi_1(V) = 1$. In particular, we get that if $\pi_1(U)$ and $\pi_1(V)$ are both trivial, then $\pi_1(X)$ is trivial. This leads to the following example.

Proposition 2.3.5.

We have $\pi_1(S^n) = 1$ for all $n \geq 2$.

▶ *Proof.* Decompose S^n into two open discs U and V, so $U \cap V \simeq [0,1] \times S^{n-1}$.



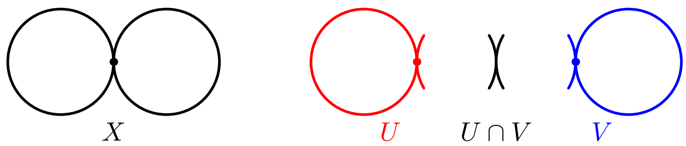
Since $n \geq 2$, we have that $U \cap V$ is path-connected. Then since $\pi_1(U) = \pi_1(V) = 1$, we get by Seifert-van Kampen theorem (Theorem 2.3.3) that $\pi_1(S^n) = 1$. $\square$

Thus, we have establish that $\pi_1(S^2) = 1$ and $\pi_1(\text{annulus}) = \mathbb{Z}$, as informally described at the beginning of this chapter, showing that these two spaces are not homeomorphic.

The next example presents the next simplest case, when the two groups $\pi_1(U)$ and $\pi_1(V)$ are nontrivial, but $\pi_1(U \cap V) = 1$. In this case, $\pi_1(X)$ will be just be the free product.

Example 2.3.6 ($S^1 \vee S^1$)**.**

We compute $\pi_1(S^1 \vee S^1)$. To do this, we cut $S^1 \vee S^1$ into U and V, where U and V each contain one circle and meet in the middle. (The reason for doing this is to make sure that U and V are open and overlap.)



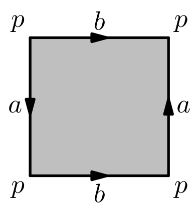
Since U and V are both homotopy equivalent to S^1 (can you convince yourself this?), we get that $\pi_1(U) = \pi_1(V) = \mathbb{Z}$. However, $U \cap V$ is homotopy equivalent to a point, so $\pi_1(U \cap V) = 0$. Thus, by Seifert-van Kampen's theorem, we get that $\pi_1(X) = \mathbb{Z} \amalg \mathbb{Z} = \boxed{F_2}$, the free group generated by two elements.

Similarly, we can show that the wedge sum n copies of S^1 has fundamental group F_n, the free group generated by n elements.

Next, we do an example where all three groups are nontrivial.

Example 2.3.7 (Klein bottle)**.**

We compute $\pi_1(K)$, where K is the Klein bottle. Recall that the Klein bottle is a quotient of the square obtained by identifying the two edges labeled a together and the two edges labeled b together.



We make the following setup:

- Let U be the interior of the square, so U is homotopy equivalent to D^2
- Let V be homotopy equivalent to image of the boundary of the square under quotient projection. Since all the vertices of the square are identified to be the same, we get that V is homotopy equivalent to two circles touching at a point (as in the previous example).
- $U \cap V$ is homotopy equivalent to S^1, being a loop passing through a, b, a, and b^{-1}.

Thus, we have the pushout diagram

$$\begin{tikzcd} \mathbb{Z} \arrow[r] \arrow[d, "i"] & 1 \arrow[d] \\ F_2 \arrow[r] & \pi_1(K) \end{tikzcd}.$$

Thus, we have that $\pi_1(K)$ is a quotient of F_2 with the image of i. Suppose that the generators of F_2 are a and b, where a and b follows the direction in the image above. Then we have that $i(1) = abab^{-1}$. Therefore, we deduce that

$$\pi_1(K) = \boxed{\langle a, b \mid abab^{-1} = 1\rangle}.$$

✍ **Exercise 2.3.8.** Prove that $\pi_1(\mathbb{RP}^2) = \mathbb{F}_2$.

§2.3.3 Proof Sketch

The direct proof of this theorem is long and not very enlighthening, so we sketch it and refer the reader to [Hat02, Thm. 1.20] for the full details. Once we learn about covering spaces, we will give a cleaner proof in Section 3.3.2 assuming some mild conditions on X.

First, we note that the inclusion map $U \to X$ and $V \to X$ gives the maps $\pi_1(U) \to X$ and $\pi_1(V) \to X$. Therefore, we have a map $\phi : \pi_1(U) \amalg \pi_1(V) \to \pi_1(X)$. It then suffices to show that

(a) ϕ is surjective; and

(b) $\operatorname{Ker}\phi = \langle \pi(i_1)(a)\,\pi(i_2)(a)^{-1} : a \in \pi_1(U \cap V)\rangle$.

We first prove (a). To do this, we have to show that any loop in X is a product of loops in U and in V. Intuitively, this is clear because a loop in X can be cut into parts that are in U and parts that are in V. The only issue that remains is that the number of parts have to be finite. To do this, compactness comes into rescue.

Lemma 2.3.9.

Given any loop f in X, there exists a subdivision $0 = s_0 < s_1 < \cdots < s_n < s_{n+1} = 1$ such that $f([s_i, s_{i+1}])$ is contained in either U or V for all $i = 0, 1, \dots, n$.

▶ *Proof.* For each $t \in [0, 1]$, if $f(t) \in U$, take an open interval $I_t \ni t$ such that $f(I_t) \subseteq U$, and if $f(t) \in V$, take an open interval $I_t \ni t$ such that $f(I_t) \subseteq V$. The set $\{I_t : t \in [0, 1]\}$ form an open cover of $[0, 1]$, so there is a finite subcover $I_{t_0}, I_{t_1}, \dots, I_{t_{n+1}}$, where $t_0 < t_1 < \cdots < t_{n+1}$. Clearly, $t_0 = 0$ (otherwise, 0 is not in this cover) and similarly, $t_{n+1} = 1$.

Thus, for each i, there let $s_i \in I_{t_i} \cap I_{t_{i+1}}$. Then we have that $[s_i, s_{i+1}] \subseteq I_{t_{i+1}}$, which means that $f([s_i, s_{i+1}])$ is contained in either U or V. Therefore, $s_0, s_1, \dots, s_{n+1}$ meet the requirement. □

▶ *Proof of (a).* Pick any $f \in \pi_1(X)$, and let p be any point in $U \cap V$. Take a subdivision $0 = s_0 < s_1 < \cdots < s_n < s_{n+1} = 1$ as in the previous lemma. Let γ_i be a path from p to $f(s_i)$, and define We let g_i be the loop defined by following γ_i, $f([s_i, s_{i+1}])$, and then $\overline{\gamma}_i$. Thus, g_i is in the image of either $\pi_1(U)$ or $\pi_1(V)$. Finally, f is homotopic to $g_0 g_1 \dots g_n$ because that loop goes from

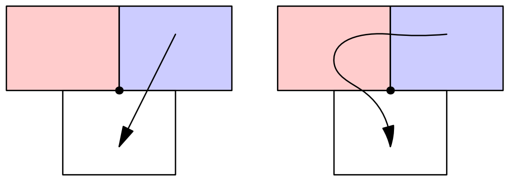
$$p \to f(s_0) \to f(s_1) \to p \to f(s_1) \to f(s_2) \to p \to \dots,$$

and the part where $f(s_i) \to p \to f(s_i)$ is the concatenation of γ_i and $\overline{\gamma}_i$, which is nulhomotopic. Thus, we cancel out parts of the form $f(s_i) \to p \to f(s_i)$, which gives that $g_0 g_1 \dots g_n$ is homotopic to $f(s_0) \to f(s_1) \to \cdots \to f(s_{n+1})$, which is equal to f. □

The fun part of the proof is (b). It is clear that for any $a \in \pi_1(U \cap V)$, we have that $\pi(i_1)(a)\,\pi(i_2)(a)^{-1}$ is in the kernel of ϕ. The hard part is to show that the kernel is contained in the normal subgroup generated by those elements.

▶ *Proof Sketch of (b).* Suppose that a loop $f = g_1 h_1 g_2 h_2 \cdots g_n h_n$ is in $\operatorname{Ker} \phi$, where g_i is in the image of $\pi_1(U)$ and h_i is in the image of $\pi_1(V)$. Take a homotopy $H : [0,1]^2 \to X$ to the trivial loop. By the compactness argument similar to Lemma 2.3.9, we can subdivide $[0,1]^2$ into a grid such that the image of each cell in the grid when going through H is contained in either U or V. We color a cell red if the image is contained in U and blue of the image is contained in V. We perturb the vertical edges in the grid so that each vertex is an intersection of at most 3 cells (not 4).

Let's say that we start with the right edge, which is the trivial loop. We want to keep inserting multiples of elements of the form $\pi_1(i_1)(a)\,\pi_1(i_2)(a)^{-1}$ to move the path to the left side. To do this, let's consider a vertex where the path pass through from the right, like the left figure below.



We claim that we can insert an element of that form to redirect the path around that vertex so that it looks like the right. To do that, suppose that the cells are colored like in the above picture. We take a small loop a around that vertex that is contained in $U \cap V$. Then we insert $\pi_1(i_1)(a)\,\pi_1(i_2)(a)^{-1}$ in the part around that vertex to redirect the path as desired.

We keep redirecting the path until all vertices are pushed to the right of the path. Eventually, we end up with a path that coincide with f, which implies that f lies in the normal subgroup generated by $\pi_1(i_1)(a)\,\pi_1(i_2)(a)^{-1}$. □

§2.4 Problems

Problem 2.A. The **infinite-dimensional sphere** S^∞ is a chain complex defined as the union of

$$S^0 \subseteq S^1 \subseteq S^2 \subseteq S^3 \dots,$$

where S^n is obtained by attaching two cells of onto S^{n-1}. Prove that S^∞ is homotopy equivalent to a point!

Problem 2.B. Prove that the Möbius band is homotopy equivalent to the cylinder $S^1 \times [0,1]$, but they are not homeomorphic.

(Hint for the latter: remove one point.)

Problem 2.C. (a) (Eckmann-Hilton) Suppose that A is a set with two unital multiplications $(\times_1, u_1)$ and $(\times_2, u_2)$ such that for all $a, b, c, d \in A$, we have

$$(a \times_1 b) \times_2 (c \times_1 d) = (a \times_2 c) \times_1 (b \times_1 d).$$

Prove that the $\times_1$ and $\times_2$ are equal, associative, and commutative.

(b) Let G be a path-connected topological group (a group and a topological space such that multiplication map $G \times G \to G$ and inverse map $G \to G$ are continuous). Prove that $\pi_1(G)$ is abelian.

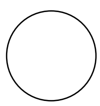
Problem 2.D. Compute the fundamental group of the following spaces

(a) the g-holed torus Σ_g.

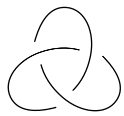
(b) the g-holed torus Σ_g with b points removed.

(c) the connected sum $\mathbb{RP}^2 \# \mathbb{RP}^2 \# \mathbb{RP}^2$.

(d) $\mathbb{R}^3 \setminus S^1$, where S^1 is embedded as an unknot as shown below:



(e) $\mathbb{R}^3 \setminus S^1$, where S^1 is embedded as a trefoil knot as shown below:



(f) $S^1 \times [0,1]/\sim$, where we view S^1 as the unit circle $\{z \in \mathbb{C} : |z| = 1\}$ and the relation $\sim$ identifies $(e^{it}, 0)$ with $(e^{5it}, 1)$ for all $t \in [0, 2\pi]$.

Problem 2.E. Prove that

$$\pi_1(\mathbb{RP}^n) = \begin{cases} \mathbb{Z} & \text{if } n = 1 \\ \mathbb{Z}/2 & \text{if } n \geq 2 \end{cases} \quad \text{and} \quad \pi_1(\mathbb{CP}^n) = 1$$

for all positive integer n.

3 Covering Spaces

We have seen one example of covering space when we explained the (currently incomplete) proof of Theorem 2.2.12 that $\pi_1(S^1) = \mathbb{Z}$, where we utilize the map $\mathbb{R} \to S^1$ that wraps the real line into a circle. This is an example of covering spaces, where here, $\mathbb{R}$ is a cover of S^1. We will make this notion precise and complete this proof.

Covering spaces are nice because we can classify them in terms of the fundamental group. More specifically, all connected covering spaces of X correspond to subgroups of $\pi_1(X)$. This allows us to translate back and forth between algebra aspects of $\pi_1(X)$ and geometric aspects of the covering spaces easily. Later, we will see some group-theoretic applications and the clean proof of van Kampen theorem.

§3.1 Definition and Basic Properties

In this section, we define covering spaces and prove lifting property, which will allow us to complete the proof of Theorem 2.2.12.

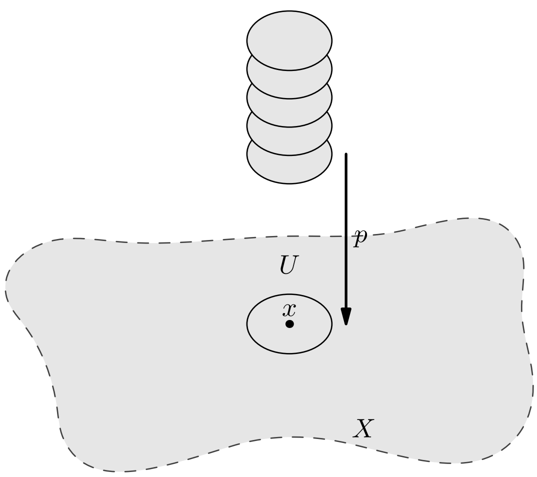
§3.1.1 Definition

Definition 3.1.1.

A **covering** of a topological space X is a topological space $\widetilde{X}$ and a surjective map $p : \widetilde{X} \to X$ such that for each point $x \in X$, there exists an open neighborhood U of X such that

- $p^{-1}U = \coprod_{i \in I} U_i$ is a disjoint union of copies of U; and
- the restriction $p|_{U_i}$ induces a homeomorphism $U_i \to U$.

Informally, the space $\widetilde{X}$ lies above X. Locally, $\widetilde{X}$ looks like a disjoint union of copies of X, but globally, this might not be true. Here is a picture that depicts a way to think about covering spaces.



Covering spaces form a category.

Definition 3.1.2.

For any topological space X, the category $\mathrm{Cov}(X)$ is a category with

- objects being all covering spaces of X;
- morphisms from covering spaces $p_1 : \widetilde{X}_1 \to X$ to $p_2 : \widetilde{X}_2 \to X$ are described by continuous maps $f : \widetilde{X}_1 \to \widetilde{X}_2$ such that the following diagram commutes:

$$\begin{tikzcd} \widetilde{X}_1 \arrow[rr, "f"] \arrow[dr, "p_1"'] & & \widetilde{X}_2 \arrow[dl, "p_2"] \\ & X & \end{tikzcd}$$

Our goal is to classify all covering spaces of X up to isomorphism.

§3.1.2 Examples

We now do a large number of examples. First, a trivial one.

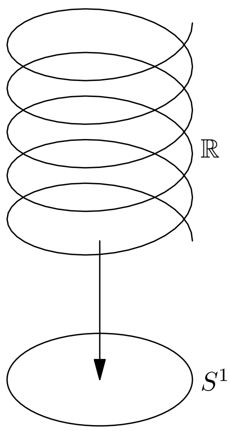
Example 3.1.3 (Trivial example).

A disjoint union of copies of X, $\coprod_{i \in I} X$, is a covering space of X.

Example 3.1.4 (Covering spaces of S^1).

We now describe all connected covering spaces of S^1, although we will prove later that these are all such coverings. We view S^1 as the complex unit circle $\{z \in \mathbb{C} : |z| = 1\}$.

- First, the map $\mathbb{R} \to S^1$ given by $t \mapsto e^{2\pi i t}$ makes $\mathbb{R}$ a covering space of S^1.



- For each $n = 1, 2, 3, \ldots$, the map $S^1 \to S^1$ given by $z \mapsto z^n$ (wrapping around the circle n times) makes S^1 an n-fold cover of S^1.

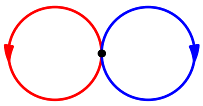
Example 3.1.5 (Covering spaces of $\mathbb{RP}^n$).

It turns out that there are two connected covers of $\mathbb{RP}^n$: the trivial cover $\mathbb{RP}^n \to \mathbb{RP}^n$ and the quotient projection $S^n \to \mathbb{RP}^n$.

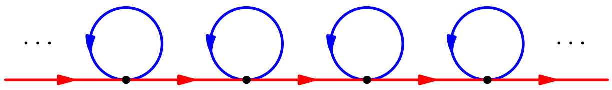
Example 3.1.6 (Some covering spaces of $S^1 \times S^1$).

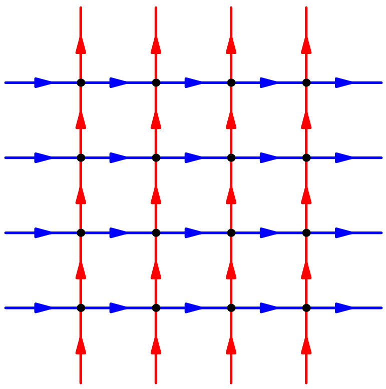
Product of covering spaces is a covering space. For example $\mathbb{R} \times \mathbb{R}$, $\mathbb{R} \times S^1$, and $S^1 \times \mathbb{R}$ can all be realized as a covering space of $S^1 \times S^1$.

▶ **Extended example: some covering spaces of $S^1 \vee S^1$.** There are numerous covering spaces of $S^1 \vee S^1$. We draw $S^1 \vee S^1$ with red edge, blue edge, and black dot as follows:



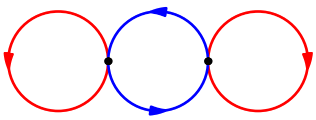
Some easier examples of covering spaces of $S^1 \vee S^1$ can be obtained by unwrapping one or both of the circles.





Here, the color of the edges indicate where the covering map goes. All black dots goes to the black dots in $S^1 \vee S^1$, while red and blue edges correspond to red and blue circle, respectively.

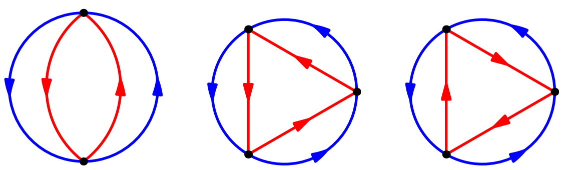
These are far from all examples. One less obvious example is the following:

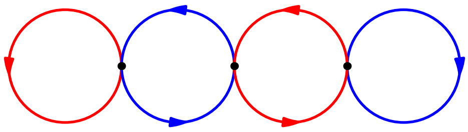


This is a covering space because at each black dot, there are exactly four edges:

- one outgoing red edge;
- one incoming red edge;
- one outgoing blue edge; and
- one incoming blue edge.

Once we know this, we can draw numerous more examples.





✍ **Exercise 3.1.7.** Draw a cover of $S^1 \vee S^1$ that is not shown above.

§3.1.3 Lifting Property

We now state the lifting property of covering spaces. First, we state it in the most general form.

Theorem 3.1.8 (Lifting Property of Covering Spaces)**.**

Given a covering space $p : \widetilde{X} \to X$, a map $h : [0,1] \times Y \to X$, and $\widetilde{h}_0 : Y \to \widetilde{X}$ such that $p(\widetilde{h}_0(y)) = h(0, y)$ for all $y \in Y$.

Then there exists unique **lift** $\widetilde{h} : [0,1] \times Y \to \widetilde{X}$ such that $\widetilde{h}(0, \bullet) = \widetilde{h}_0$ and $p \circ \widetilde{h} = h$.

Before proving this, we discuss some consequences. Plugging in $Y = \{\text{pt}\}$ gives the following corollary.

Corollary 3.1.9 (Path Lifting)**.**

Given a covering space $p : \widetilde{X} \to X$, a path $\gamma : [0,1] \to X$, and a point $\widetilde{x}_0 \in p^{-1}(\gamma(0))$. Then there exists unique map $\widetilde{\gamma} : [0,1] \to \widetilde{X}$ and $p \circ \widetilde{\gamma} = \gamma$.

(In other words, given a path in X and a point $\widetilde{x}_0$ upstairs, the path lifts uniquely to a path in $\widetilde{X}$ from x_0. Notably, if γ is a loop, then $\widetilde{\gamma}$ need not be a loop, since we could end up with a different preimage of $\gamma(0)$.)

Furthermore, plugging in $Y = [0,1]$ and $\widetilde{h}_0$ being a constant map to some point $\widetilde{x}_0$, we get the following corollary.

Corollary 3.1.10.

For any homotopy $h : [0,1]^2 \to X$ with $h(0, s) = x_0$ for all s and any $\widetilde{x}_0 = p^{-1}(x_0)$, there exists unique map $\widetilde{h} : [0,1]^2 \to \widetilde{X}$ such that $\widetilde{h}(0, s) = \widetilde{x}_0$ for all s and $p \circ \widetilde{h} = h$.

Using the above two corollaries with $\widetilde{X} = \mathbb{R}$ and $X = S^1$ completes the proof of Theorem 2.2.12.

We now prove Theorem 3.1.8. The idea is that we use the definition of covering spaces to see that the map is well-defined locally. Then we paste those maps together to get a globally-defined lift. The execution of the proof is slightly tedious, since we have to use compactness argument to show that there are finitely many pieces to be defined locally. It might be helpful to think about the case where $Y = \{\text{pt}\}$ first, where some of the complication of the proof is removed.

▶ *Proof of Theorem 3.1.8.* The proof proceeds in two steps.

(a) We prove that for any $y_0 \in Y$, there exists an open neighborhood $y_0 \in V$ such that $\widetilde{h}$ is defined on $[0,1] \times V$.

(b) We prove that $\widetilde{h}$ is unique, and it suffices to do this when the case where $Y = \{\text{pt}\}$.

These two steps complete the proof, since by (b), the map obtained from different points in Y will agree at the intersection, so pasting lemma (Problem 1.C) tells that we can glue these maps together. This completes the proof of existence.

Now, we proceed through these steps.

(a) We do the compactness argument. For each $t \in [0,1]$, let $U_t \subset X$ be an open neighborhood around $h(t, y_0)$ such that the preimage of U_t is a disjoint union of copies of U_t. Let $I_t \subseteq [0,1]$ be an open interval such that $I_t \times \{y_0\} \subseteq h^{-1}U_t$.

Since $t \in I_t$, we get that $(I_t)_{t\in[0,1]}$ is an open cover of $[0,1]$. Thus, there exists a finite subcover $I_{t_1}, \ldots, I_{t_n}$ where $0 \le t_1 < \cdots < t_n \le 1$. For each i, there exists an open neighborhood $y_0 \in V_i \subseteq Y$ such that $I_{t_i} \times V_i \subseteq h^{-1}U_t$. We now take

$$V = \bigcap_{i=1}^{n} V_i.$$

To see why this works, we inductively define the map on $(I_{t_1} \cup \cdots \cup I_{t_i}) \times V$ for each $i = 1, 2, \ldots$. The base case $i = 1$ is clear, since we can make the image $\widetilde{h}(I_{t_1} \times V)$ sit on the copy of U_{t_1} that $\widetilde{h}(0, y_0)$ resides on by composing h with the inverse of p on that copy.

For the inductive step, suppose that we have defined up to $(I_{t_1} \cup \cdots \cup I_{t_i}) \times V$. Let $W = (U_{t_1} \cup \cdots \cup U_{t_i}) \cap U_{t_{i+1}}$. The preimage $p^{-1}W$ is a disjoint union of copies of W. Select a copy W' that is contained in the copy of U_{t_i} that $h(t_i, y_0)$ resides in. Then we select the copy of $U_{t_{i+1}}$ that contains W'. By mapping everything in $I_{t_{i+1}} \times V$ to that copy of $U_{t_{i+1}}$, we get a map $I_{t_{i+1}} \times V$ that agrees on the previous maps on $((I_{t_1} \cup \cdots \cup I_{t_i}) \cap I_{t_{i+1}}) \times V$ (since $h((I_{t_1} \cup \cdots \cup I_{t_i}) \cap I_{t_{i+1}}) \subseteq W$). Thus, it can be pasted into the previous map by the pasting lemma Problem 1.C.

(b) We write the map as $h : [0,1] \to X$. Again, by compactness, there exists open intervals $I_{t_1}, \ldots, I_{t_n}$ such that $I_{t_i} \subseteq h^{-1}U_i$ for some neighborhood U_i around $h(t_i)$ such that preimage of U_i is a disjoint union of copies of U_i.

Now, suppose that $\widetilde{h}_1$ and $\widetilde{h}_2$ are two lifts. We inductively show that $\widetilde{h}_1|_{I_{t_i}} = \widetilde{h}_2|_{I_{t_i}}$. For the base case $i = 1$, note that $\widetilde{h}_1(I_{t_1})$ and $\widetilde{h}_2(I_{t_1})$ each is in one of the copies of U_i. They must be in the same copy since they agree at 0. Thus, by composing with the inverse of p, we get that $\widetilde{h}_1|_{I_{t_1}} = \widetilde{h}_2|_{I_{t_1}}$.

The inductive step is pretty similar. Suppose we know that $\widetilde{h}_1|_{I_{t_{i-1}}} = \widetilde{h}_2|_{I_{t_{i-1}}}$. Since $h(I_{t_i}) \subseteq U_i$, we get that $\widetilde{h}_1(I_{t_i})$ and $\widetilde{h}_2(I_{t_i})$ are in one of the copies of U_i. They must be in the same copies since they agree on $U_{i-1} \cap U_i$ (which is nonempty since $I_{t_{i-1}} \cap I_{t_i}$ is nonempty). (Here we used the inductive hypothesis that $\widetilde{h}_1|_{I_{t_{i-1}}} = \widetilde{h}_2|_{I_{t_{i-1}}}$.) Thus, by composing with the inverse of p, we get that $\widetilde{h}_1|_{I_{t_i}} = \widetilde{h}_2|_{I_{t_i}}$. □

§3.2 Classification Theorem for Covering Spaces

§3.2.1 The Correspondence

Given a topological space X and basepoint $x_0 \in X$, we will provide a bijection between

$$\left\{ \begin{array}{c} \text{covering spaces} \\ p : (\widetilde{X}, \widetilde{x}_0) \to (X, x_0) \end{array} \right\} \longrightarrow \{\text{subgroups of } \pi_1(X, x_0)\}.$$

(The map $p : (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ is a covering space if $p : \widetilde{X} \to X$ is a covering map and $p(\widetilde{x}_0) = x_0$.)

One direction of the correspondence is more apparent: we just take the image of the map

$$\pi_1(p) : \pi_1(\widetilde{X}, \widetilde{x}_0) \to \pi_1(X, x_0).$$

We note the following useful fact.

Proposition 3.2.1.

Given a covering space $p : (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$, the map $\pi_1(p)$ is injective.

▶ *Proof.* Suppose that there exists a loop $\gamma : [0,1] \to \widetilde{X}$ based at $\widetilde{x}_0$ such that $p \circ \gamma$ is nulhomotopic. In particular, there exists homotopy $h : [0,1] \times [0,1] \to X$ such that $h(t,0) = h(t,1) = x_0$, $h(0,t) = p(\gamma(t))$, and $h(1,t) = x_0$ for all t.

By lifting property, let $\widetilde{h} : [0,1] \times [0,1] \to \widetilde{X}$ be a lift of h such that $h(0,t) = \gamma(t)$. for all t. Moreover, by uniqueness of path lifting lift, $\widetilde{h}(t,0) = \widetilde{h}(t,1) = \widetilde{h}(1,t) = \widetilde{x}_0$ for all t. Thus, $\widetilde{h}$ is a homotopy from γ to a constant loop. □

The classification theorem of covering spaces only works under some mild assumptions. We state precisely what those assumptions are, and then we will mostly forget about them.

Definition 3.2.2.

A topological space X is **nice** if it is

- **path-connected** (any two points in X is connected by a path);
- **locally path-connected** (for any point x and open set $U \ni x$, there exists a path-connected open set V such that $x \in V \subseteq U$, as defined in Problem 1.E);
- **semi-locally simply-connected**, defined as for every $x \in X$, there exists a neighborhood $x \in U \subseteq X$ such that every loop in $\pi_1(U,x)$ maps to a trivial loop in $\pi_1(X,x)$.

In the next two subsections, we will prove the following theorem.

Theorem 3.2.3 (Classification of Covering Spaces)**.**

Let X be a nice topological space and $x_0 \in X$ be a basepoint. Then the map

$$\left\{ \begin{array}{c} \text{connected covering spaces (up to isomorphism)} \\ p : (\widetilde{X}, \widetilde{x}_0) \to (X, x_0) \end{array} \right\} \longrightarrow \{\text{subgroups of } \pi_1(X,x_0)\}.$$

given by taking $p : (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ to the subgroup $\pi_1(p)\big(\pi_1(\widetilde{X}, \widetilde{x}_0)\big)$ is a bijection.

The proof of this theorem is fairly complex, but it is as interesting as the statement itself. We will spend the next two sections discussing the proof. For now, let us illustrate the theorem with several examples.

▶ Examples

Example 3.2.4 (Covers of a simply-connected space)**.**

If X is nice and simply-connected, then the only connected covering space of X is the trivial one.

Example 3.2.5 (All covers of S^1)**.**

We can now show that the covers given in Example 3.1.4 are all possible connected covers of S^1.

The subgroups of $\pi_1(S^1) = \mathbb{Z}$ are $\{0\}$ and $n\mathbb{Z}$ for some positive integer n. We note the following.

- The first cover $\mathbb{R} \to S^1$ corresponds to the subgroup $\{0\}$, since $\mathbb{R}$ is simply connected.
- The cover $p : S^1 \to S^1$ given by the map $z \mapsto z^n$ corresponds to the subgroup $n\mathbb{Z}$, since a single loop in the left S^1 maps through p to a loop that winds around S^1 n times, which is an element $n \in \pi_1(S^1)$. Thus, the image is the subgroup generated by n, or

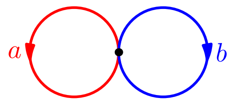
$n\mathbb{Z}$.

The classification theorem then tells us that these are all possible connected covering spaces of S^1.

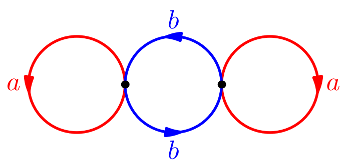
Analogously, we can show that the covering spaces in Example 3.1.5 are all possible covers of $\mathbb{RP}^n$.

Example 3.2.6.

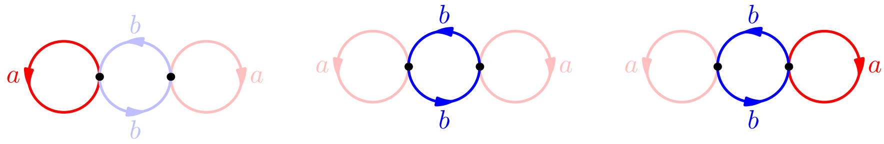
Recall that $\pi_1(S^1 \vee S^1)$ (with black dot as the basepoint) is the free group generated by a and b shown below:



Since there are numerous subgroups of F_2, there are numerous covering spaces of $S^1 \vee S^1$, some of which are shown previously. Let us find the subgroup corresponding to the following cover X.



First, we have to pick the basepoint. Let us use the left dot as the basepoint. Then $\pi_1(X)$ is a free group with three generators. The generators are as follows.



a $\quad$ b^2 $\quad$ $ba^{-1}b$

Thus, this cover corresponds to the subgroup

$$\langle a, b^2, ba^{-1}b \rangle \subseteq F_2.$$

Thus, we have found a way to realize F_3 as a subgroup F_2! It's not completely obvious why this subgroup is free. What's even more surprising is that this subgroup has index 2, and it follows from the fact that this covering map is 2-to-1 (we leave this detail to the reader). This hints at how this theorem have found group-theoretic applications.

▶ The Universal Cover

Definition 3.2.7.

The **universal cover** of a nice topological space X is the covering space corresponds to the trivial subgroup. It is the unique simply-connected cover of X.

One of the cruxes of the proof of the classification theorem is to show that universal covers exist. Unfortunately, the construction of a universal cover in that proof is impractical for simple spaces. Usually, constructing a universal cover of a specific topological spaces is an ad-hoc process.

Example 3.2.8.

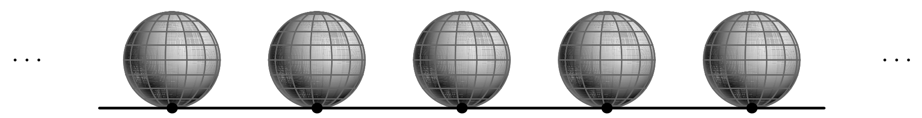
Here are some easy examples of universal covers.

- The universal cover of S^1 is $\mathbb{R}$.
- The universal cover of $\mathbb{RP}^n$ is S^n.
- The universal cover of the torus $S^1 \times S^1$ is $\mathbb{R}^2$.

Let us now do some more interesting examples of universal covers.

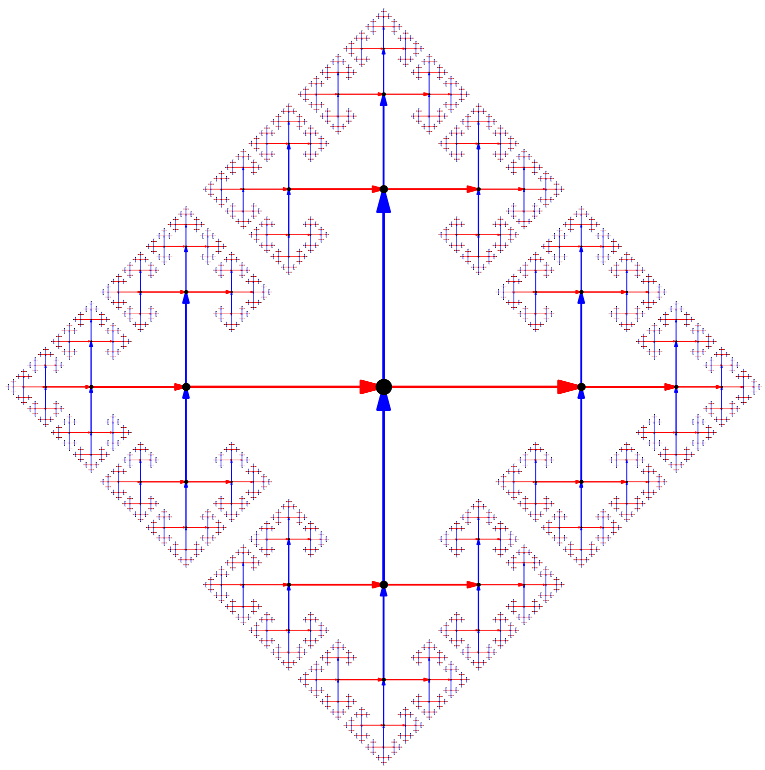
Example 3.2.9 (Universal cover of $S^1 \vee S^2$).

Note that $\pi_1(S^1 \vee S^2) = \mathbb{Z}$, and the generator comes from S^1. Thus, the universal cover of $S^1 \vee S^2$ comes from unwrapping the S^1 part, creating infinitely many copies of S^2 at each preimage. It is depicted below.



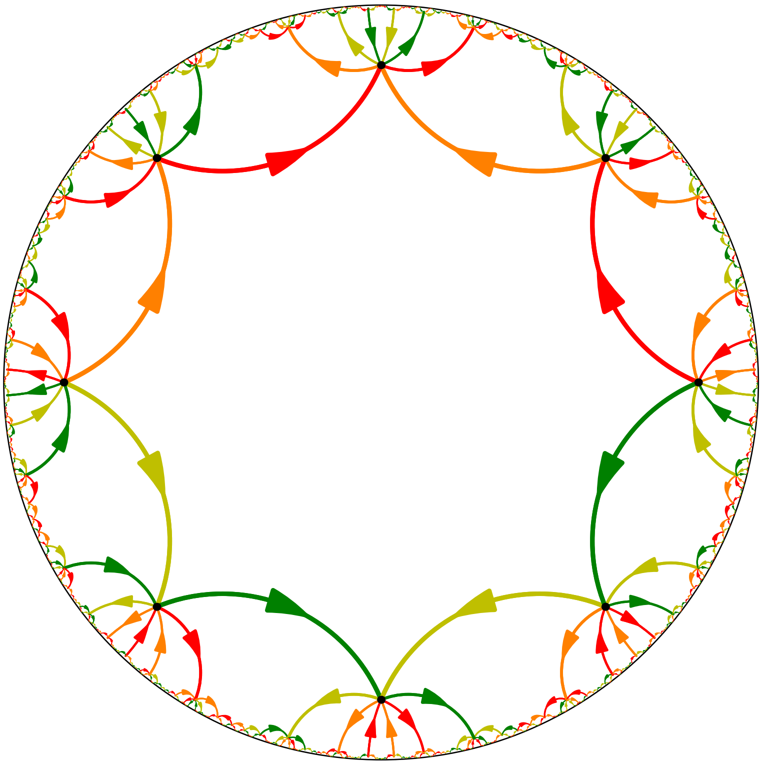
Example 3.2.10 (Universal cover of $S^1 \vee S^1$).

Recall that any covering space of $S^1 \vee S^1$ looks like a graph, with each vertex has exactly four edges, two incoming and two outgoing. The universal cover of $S^1 \vee S^1$ looks like that, with each vertex keep branching out to the four different directions. It is depicted below.



Example 3.2.11 (Universal Cover of Σ_g).

The universal cover of Σ_g (the g-holed torus) turns out to be $\mathbb{R}^2$ for all $g \geq 1$. Here is a visualization for $g = 2$ inspired by some hyperbolic geometry.



§3.2.2 Covers that induces the same subgroups are isomorphic

We now begin proving Theorem 3.2.3. In this subsection, we prove the injectivity part of the theorem: that any two covers with the same subgroups are isomorphic. Proving this requires the following nice lemma.

Lemma 3.2.12 (General Lifting Criterion).

Let X and Y be nice topological spaces with $x_0 \in X$ and $y_0 \in Y$, and let $p : (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ be a connected covering space. Then a map $f : (Y, y_0) \to (X, x_0)$ lifts to a map $\widetilde{f} : (Y, y_0) \to (\widetilde{X}, \widetilde{x}_0)$ (i.e., there exists $\widetilde{f}$ satisfying $p \circ \widetilde{f} = f$) if and only if the image of $\pi_1(f)$ is contained in the image of $\pi_1(p)$.

Moreover, if such lift exists, it is unique.

This lemma is powerful because it reduces the geometric question of finding lifts to a completely algebraic question.

▶ *Proof.* **Criterion for Existence.** The "only if" direction is pretty clear, since $\mathrm{Im}(\pi_1(f)) = \mathrm{Im}(\pi_1(p \circ \widetilde{f})) \subseteq \mathrm{Im}(\pi_1(p))$.

To prove the "if" direction, we define the map using path lifting. For any point $y \in Y$, take a path $\gamma : [0, 1] \to Y$ from y_0 to y. Thus, $f \circ \gamma$ is a path from x_0 to $f(y)$, which then lifts to a path $\widetilde{f \circ \gamma}$. Define $\widetilde{f}(y) = \widetilde{f \circ \gamma}(1)$.

We have to show that this does not depend on γ. To see this, suppose that γ_1 and γ_2 are two paths from y_0 to y. Concatenate them to get a loop $\gamma_1\overline{\gamma}_2$ around y_0, and so we get a loop $f \circ \gamma_1\overline{\gamma}_2$

around x_0. This loop is in the image of $\pi_1(f)$, so it must be in form $p\circ\delta$ for some loop $\delta \in \pi_1(\widetilde{X}, \widetilde{x}_0)$. In particular, $f \circ \gamma_1\overline{\gamma}_2$ lifts to δ (by uniqueness of path lifting), and in particular, it lifts to a loop. Comparing both endpoints of this loop gives $\widetilde{f \circ \gamma_1}(1) = \widetilde{f \circ \gamma_2}(1)$, so $\widetilde{f}(y)$ does not depend on the choice of γ.

We have to show that $\widetilde{f}$ is continuous. (The reader should feel free to skip this, but note that it uses the fact that Y is locally path-connected.) Consider an open set $V \subseteq \widetilde{X}$. By shrinking V, it suffices to show that $\widetilde{f}^{-1}U$ is open in the case where p induces a homeomorphism from V to some open subset $U \subseteq X$. Since $f^{-1}U$ is open and Y is locally path-connected, we can define $\widetilde{f}(y)$ for any $y \in f^{-1}U$ by using a path contained in $f^{-1}U$, and thus $\widetilde{f}(y) \subseteq V$. Thus, $f^{-1}U \subseteq \widetilde{f}^{-1}V$

On the other hand, for any $y \in \widetilde{f}^{-1}V$, we have that $f(y) = p(\widetilde{f}(y)) \in U$, so $\widetilde{f}^{-1}V \subseteq f^{-1}U$. Combining with the previous paragraph gives $\widetilde{f}^{-1}V = f^{-1}U$. This shows that $\widetilde{f}^{-1}V$ is open.

Uniqueness. Finally, we show that lifts are unique. Suppose that $\widetilde{f}_1$ and $\widetilde{f}_2$ are two such lifts. Take any path $\gamma : [0,1] \to Y$ from y_0 to y. Then we note that $\widetilde{f}_1 \circ \gamma$ and $\widetilde{f}_2 \circ \gamma$ is a path that both lift $f \circ \gamma$. Thus, by uniqueness of lift, we have $\widetilde{f}_1 \circ \gamma = \widetilde{f}_2 \circ \gamma$, so comparing the endpoint gives $f_1(y) = f_2(y)$. □

With the general lifting condition, the proof of injectivity in the classification theorem (Theorem 3.2.3) is fairly short.

▶ *Proof of injectivity in Theorem 3.2.3.* Suppose that $p_1 : (\widetilde{X}_1, \widetilde{x}_1) \to (X, x_0)$ and $p_2 : (\widetilde{X}_2, \widetilde{x}_2) \to (X, x_0)$ are such that $\operatorname{Im}(\pi_1(p_1)) = \operatorname{Im}(\pi_1(p_2))$. Then by applying the lifting criterion, Lemma 3.2.12, we find that the map p_1 lifts to $\widetilde{p}_1 : (\widetilde{X}_1, \widetilde{x}_1) \to (\widetilde{X}_2, \widetilde{x}_2)$ and p_2 lifts to $\widetilde{p}_2 : (\widetilde{X}_2, \widetilde{x}_2) \to (\widetilde{X}_1, \widetilde{x}_1)$.

We claim that $\widetilde{p}_1$ and $\widetilde{p}_2$ are inverses. To see this, note that $\widetilde{p}_1 \circ \widetilde{p}_2$ is a lift of identity map since $\widetilde{p}_1 \circ \widetilde{p}_2 \circ p_1 = \widetilde{p}_1 \circ p_2 = p_1 = p_1 \circ \mathrm{id}$. However, the (unique) lift of identity map is identity map, so we have that $\widetilde{p}_1 \circ \widetilde{p}_2 = \mathrm{id}$. Similarly, $\widetilde{p}_2 \circ \widetilde{p}_1 = \mathrm{id}$, proving that $\widetilde{p}_1$ and $\widetilde{p}_2$ are inverses. □

§3.2.3 There exists a cover for each subgroup

We now prove the surjectivity part of Theorem 3.2.3. Given a subgroup $H \le \pi_1(X, x_0)$, we have to construct a covering space $p : (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ such that $\operatorname{Im}(\pi_1(p)) = H$. It turns out that most of the work occurs at the case when H is trivial, i.e., the construction of the universal cover. We do that first, then we will quickly deduce the surjectivity part of Theorem 3.2.3.

▶ **Construction of the Universal Cover** Let X be a nice topological space and $x_0 \in X$. We construction the universal cover of X in three steps.

- **The Set.** As a set, the universal cover is a homotopy class of path starting from x_0:

  $$\widetilde{X} = \{[\gamma] : \gamma : [0,1] \to X, \gamma(0) = x_0\},$$

  where $[\bullet]$ denote the homotopy class: an equivalence class where two paths γ_1 and γ_2 are equivalent if there exists homotopy (that must preserve endpoints) between the two.

- **The Topology.** Intutively, we want to keep paths that are close together in the same open set. We define

  $$\mathcal{U} = \{U \subseteq X \text{ open, path-connected} : \pi_1(U) \to \pi_1(X) \text{ is trivial}\}.$$

  Now, a subset of $\widetilde{X}$ is open if and only if it is a union of elements in form

  $$U_{[\gamma]} = \{[\gamma \cdot \eta] : \eta \text{ is a path in } U \text{ such that } \eta(0) = \gamma(1)\}$$

  for some $U \in \mathcal{U}$ and $[\gamma] \in \widetilde{X}$.

- **The Covering Map.** The covering map $p : \widetilde{X} \to X$ is given by $[\gamma] \mapsto \gamma(1)$, which is well-defined since homotopy preserves endpoints.

Proposition 3.2.13.

The $\widetilde{X}$ defined above is indeed a universal cover.

▶ *Proof.* We have to check various things. The most interesting parts are the third and fourth bullet point.

- $\boldsymbol{\widetilde{X}}$ **defines a valid topology.** We have to show that open sets are closed under intersection. Specifically, given $[\gamma] \in U_{[\gamma_1]} \cap V_{[\gamma_2]}$, we have to show that there exists $W \in \mathcal{U}$ such that $W_{[\gamma]} \subseteq U_{[\gamma_1]} \cap V_{[\gamma_2]}$.
  To do this, note that
  - since X is semi-locally simply-connected, there exists $W' \subseteq U \cap V$ such that $\pi_1(W') \subseteq \pi_1(X)$ is trivial.
  - since X is locally path-connected, there exists $W \subseteq W'$ such that W is path-connected.

  Then $\pi_1(W) \to \pi_1(X)$ is trivial since it factors through $\pi_1(W')$. Thus, $W \in \mathcal{U}$. Now, given any path $[\gamma \cdot \eta] \in W_{[\gamma]}$, we have that $[\gamma \cdot \eta] = [\gamma_1 \cdot \eta_1 \cdot \eta]$ for some path η_1 contained in U, so $[\gamma \cdot \eta]$ is in $U_{[\gamma_1]}$. Similarly, $[\gamma \cdot \eta] \in V_{[\gamma_2]}$. Hence, $W_{[\gamma]} \in U_{[\gamma_1]} \cap V_{[\gamma_2]}$.
- $\boldsymbol{p}$ **is a covering map.** (Note that this automatically implies that p is continuous.) Take any $x \in X$. By repeating the same argument as above, we pick $U \in \mathcal{U}$ such that $x \in U$. Then the preimage of point x can be written as a disjoint union of sets of the form $U_{[\gamma]}$ as γ ranges across all paths from x_0 to some point in U. Note that $p|_{U_{[\gamma]}} : U_{[\gamma]} \to U$ is a bijection since any two paths that have the same endpoint can be concatenated to get a loop, which must be nulhomotopic in X, so any path in $U_{[\gamma]}$ is determined by its endpoints. By restricting to open sets, it is also not hard to show that $p|_{U_{[\gamma]}}$ is a homeomorphism. This verifies the covering space condition.
- $\boldsymbol{\widetilde{X}}$ **is path-connected.** For any $[\gamma] \in \widetilde{X}$, we define a path $\eta : [0,1] \to \widetilde{X}$ by
  $$\eta(t) = \{\text{class of path } s \mapsto \gamma(st)\}. \tag{3.1}$$
  We omit the proof that η is continuous.
- $\boldsymbol{\pi_1(\widetilde{X})}$ **is trivial.** Let e be the constant loop based at x_0. We prove that $\pi_1(\widetilde{X}, [e])$ is trivial. To do this, we show that $\operatorname{Im}(\pi_1(p))$ is trivial. Let $\gamma \in \operatorname{Im}(\pi_1(p))$. Then γ lifts to a loop at $[e]$. However, one lift of γ is the path η given in (3.1), which is a path from $[e]$ to $[\gamma]$. Thus, $[e] = [\gamma]$, so γ is nulhomotopic. □

▶ *Proof of Surjectivity Part of Theorem 3.2.3.* Let X be a nice topological space, and let Y be the universal cover. Let $H \leq \pi_1(X, x_0)$. We define the cover $\widetilde{X}$ by $\widetilde{X} = Y/\sim$, where two points $[\gamma], [\gamma'] \in Y$ are equivalent if and only if $\gamma' = \gamma \cdot \eta$ for some η such that $[\eta] \in H$ (a prerequisite to this is that γ' and γ have the same endpoints). The map $Y \to X$ descends to $p : \widetilde{X} \to X$. We omit the proof that $\sim$ is an equivalent relation.

The proof that p is a covering map is fairly similar to the proof that $Y \to X$ is a covering map above. It is tedious enough that this is not worth repeating. It is also not hard to see that $\widetilde{X}$ is path-connected.

We now show that $\operatorname{Im}(\pi_1(p)) = H$. Given $\gamma \in \pi_1(X, x_0)$, the following are equivalent:

- $\gamma \in \operatorname{Im}(\pi_1(p))$;
- γ lifts to a loop in $\widetilde{X}$;
- $[\gamma] \sim [e]$, where e is the constant loop at x_0 (this follows from that a lift of γ is given in (3.1), which is a path from $[e]$ to $[\gamma]$);
- $[\gamma] \in H$.

This concludes the proof. □

§3.2.4 The Category of Covering Spaces

We will now make a concrete description of the category $\mathrm{Cov}(X)$. Since we now know how to classify connected covering spaces of X, we only need to deal with non-connected cover. In particular, all covers of X will be just a union of connected covers of X.

Definition 3.2.14.

For any group G, the category of ***G*-set** has

- objects being a set X together with a (left) G-action;
- morphims from $X \to Y$ being all functions $f : X \to Y$ that is G-equivariant: $gf(x) = f(gx)$ for all $x \in X$.

Let X be a nice topological space. For each covering space $p : \widetilde{X} \to X$ and pick $x_0 \in X$, we define a $\pi_1(X, x_0)$-set as follows: the set is the preimage $p^{-1}(x_0)$. For each $\gamma \in \pi_1(X, x_0)$ and $\widetilde{x} \in \widetilde{X}$, we lift γ to a path from $\widetilde{x}$ to another point in $p^{-1}(x_0)$; that point is $\gamma \cdot \widetilde{x}$. Convince yourself that this is a valid group action.

This can be promoted into a functor in an obvious way: given a morphism f between covers $p_1 : \widetilde{X}_1 \to X$ and $p_2 : \widetilde{X}_2 \to X$, note that f sends an element of $p_1^{-1}(x_0)$ to an element of $p_2^{-1}(x_0)$, and thus restricting to a map of the corresponding $\pi_1(X, x_0)$-sets in an obvious way. Uniqueness of lifting shows that this map f is $\pi_1(X, x_0)$-equivariant.

Theorem 3.2.15.

Let X be a nice topological space. Then the functor $F : \mathrm{Cov}(X) \to \{\pi_1(X, x_0)\text{-sets}\}$ defined above is an equivalence of categories. In particular, it is

- **full**: the map $\mathrm{Hom}(\widetilde{X}_1, \widetilde{X}_2) \to \mathrm{Hom}(F(\widetilde{X}_1), F(\widetilde{X}_2))$ is surjective;
- **faithful**: the same map above is injective;
- **essentially surjective**: for each $\pi_1(X, x_0)$-set Y, there exists a cover $p : \widetilde{X} \to X$ such that $p^{-1}(x_0) = Y$ as $\pi_1(X, x_0)$-set.

▶ *Proof.*

- **Full and Faithful.** Let $p_1 : \widetilde{X}_1 \to X$ and $p_2 : \widetilde{X}_2 \to X$ be two covers, and let $\phi : p_1^{-1}(x_0) \to p_2^{-1}(x_0)$ be a map of $\pi_1(X, x_0)$-sets. We have to define a map $f : \widetilde{X}_1 \to \widetilde{X}_2$.

  For any point $y \in \widetilde{X}_1$, we lift a path from $p_1(y)$ to x_0 to get a path γ from y to some point $z \in p_1^{-1}(x_0)$. Then we lift the reverse of that same path to get a path from $\phi(z)$ to some point in $\widetilde{X}_1$. We let this point be $f(y)$. We can check that this continuous using an argument similar to the injectivity part Theorem 3.2.3.

  Notice that this process is canonical. In fact, uniqueness of the lift shows that f is unique.

- **Essentially Surjective.** Given a G-set Y. Note that the statement is equivalent to the classification theorem of covering spaces when Y is transitive, since the stabilizer of Y corresponds to the subgroup of that cover. If Y is a union of transitive $\pi_1(X, x_0)$-sets (i.e., orbits), we let $\widetilde{X}$ be the disjoint union of covers corresponding to each orbit. □

§3.3 Applications

§3.3.1 Subgroup of a Free Group is Free

We now explore some group-theoretic applications of covering spaces. In particular, we will prove the following theorem.

Theorem 3.3.1 (Nielsen–Schreier theorem)**.**

Every subgroup of the free group is free.

To prove this, we need some facts about covering spaces of a graph. For our purpose, a **graph** can be though as a CW complex with 0-cells are vertices, and 1-cells are edges. We note the following two facts about graphs, which we do not prove.

Proposition 3.3.2.

Any covering space of a graph must be a graph.

Intuitively, this fact follows from that a 1-cell in a graph must also correspond to a 1-cell in the cover. In general, any covering space of a CW-complex must be a CW-complex as well.

Proposition 3.3.3.

Let G be a connected graph with a spanning tree T. Then $\pi_1(G)$ is a free group generated by the edges of $G \setminus T$.

▶ *Proof Sketch.* We collapse the spanning tree T, so all vertices become a single vertex. Thus, G is homotopy equivalent to a wedge of $|G \setminus T|$ copies of S^1, which we can calculate the fundamental group using Seifert-van Kampen theorem. (If $G \setminus T$ is infinite, then we need to use the general version of Seifert-van Kampen theorem stated in Remark 2.3.4.) □

▶ *Proof of Theorem 3.3.1.* Let I be a set. We show that any subgroup the free group $G \leq F(I)$ is free.

Consider the wedge sum $X = \bigvee_{i \in I} S^1$, which has fundamental group $F(I)$. By the classification theorem of covering spaces (Theorem 3.2.3), take a cover $\widetilde{X}$ of X corresponding to the subgroup G. Then $\pi_1(\widetilde{X}) \simeq G$. However, $\widetilde{X}$ is a graph, so it must have a free fundamental group, so G is free. □

§3.3.2 Grothendieck's Proof of Seifert-van Kampen Theorem

We now present a clean and conceptual proof of Seifert-van Kampen Theorem. The major drawback is that it assumes that the spaces X, U, V, and $U \cap V$ are nice.

The idea is we want to understand geometric interpretation of morphisms from $\pi_1(X, x_0) \to G$ for arbitrary group G. The correct geometric interpretation is the following.

Definition 3.3.4.

Given any topological space X with $x_0 \in X$ and a group G, a **based regular G-cover** is a covering map $p : (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ with a continuous group action G acting on $\widetilde{X}$ such that for any $x \in X$,

- the action of G preserves the fiber $p^{-1}(x)$;
- G acts faithfully and transitively on $p^{-1}(x)$ (in other words, if we fix $\widetilde{x} \in p^{-1}(x)$, then every element of $p^{-1}(x)$ can be written uniquely as $g \cdot \widetilde{x}$ where $g \in G$.)

Proposition 3.3.5.

There exists a bijection between

$$\{\text{based regular } G\text{-covers of } X \text{ up to isomorphisms}\} \longleftrightarrow \{\text{morphisms } \pi_1(X, x_0) \to G\}.$$

▶ *Proof.* For each based regular G-cover $(\widetilde{X}, \widetilde{x}_0)$, we define $\phi : \pi_1(X, x_0) \to G$ as follows: for each $\gamma \in \pi_1(X, x_0)$, the point $\gamma \cdot \widetilde{x}_0$ (here we are considering the $\pi_1(X, x_0)$-action as in Theorem 3.2.15) can be written as $g \cdot x_0$ for some $g \in G$. We then defines $\phi(\gamma) = g$.

Injectivity. Suppose that two covers $p_1 : (X_1, \widetilde{x}_1) \to (X, x_0)$ and $p_2 : (X_2, \widetilde{x}_2) \to (X, x_0)$ induces the same map ϕ. We define a bijection $\psi : p_1^{-1}(x_0) \to p_2^{-1}(x_0)$ by sending $g\widetilde{x}_1$ to $g\widetilde{x}_2$ for all $g \in G$. For any $\gamma \in \pi_1(X, x_0)$, we have $\psi(\gamma\widetilde{x}_1) = \psi(\phi(\gamma)\widetilde{x}_1) = \phi(\gamma)\widetilde{x}_2 = \gamma\psi(\widetilde{x}_1)$, so ψ is $\pi_1(X, x_0)$-equivariant, defining a homeomorphism (by Theorem 3.2.15) $f : \widetilde{X}_1 \to \widetilde{X}_2$. This homeomorphism clearly respects the G-action, so the two covers are isomorphic.

Surjectivity. For any morphism $\phi : \pi_1(X, x_0) \to G$, we define the $\pi_1(X, x_0)$-action on G by $\gamma \cdot g = \phi(\gamma)g$. This promotes G into a $\pi_1(X, x_0)$-set, which then corresponds to a based regular G-cover. □

We now prove the Seifert-van Kampen theorem Theorem 2.3.3 under a mild assumption that U, V, and $U \cap V$ are nice.

▶ *Proof of Seifert-van Kampen theorem (Theorem 2.3.3) assuming X, U, V, and $U \cap V$ are nice.* Suppose $X = U \cup V$ where U, V, and $U \cap V$ are nice. Note that for any group G, there is a correspondence between

- morphisms $\pi_1(X, x_0) \to G$,
- based regular G-covers of (X, x_0);
- based regular G-covers $(\widetilde{U}, \widetilde{u}_0) \to (U, x_0)$, $(\widetilde{V}, \widetilde{v}_0) \to (V, x_0)$, and $(\widetilde{I}, \widetilde{i}_0) \to (U \cap V, x_0)$, together with inclusion $\widetilde{I} \to \widetilde{U}$ and $\widetilde{I} \to \widetilde{V}$ that commutes with $U \cap V \to U$ and $U \cap V \to V$.
- morphisms $\pi_1(U, x_0) \to G$, $\pi_1(V, x_0) \to G$, and $\pi_1(U \cap V, x_0) \to G$ that makes the following diagram commutes

$$\begin{tikzcd} \pi_1(U \cap V, x_0) \arrow[r] \arrow[d] \arrow[dr] & \pi_1(U, x_0) \arrow[d] \\ \pi_1(V, x_0) \arrow[r] & G \end{tikzcd}$$

This gives that $\pi_1(X, x_0)$ is the desired pushout. □

§3.4 Problems

Problem 3.A. Draw the universal covers of $\mathbb{RP}^2 \vee \mathbb{RP}^2$ and $\mathbb{RP}^2 \vee \mathbb{RP}^2 \vee \mathbb{RP}^2$.

Problem 3.B (Higher Homotopy Groups of S^1)**.** Let X be a simply-connected topological space. Prove that any continuous map $f : X \to S^1$ is nulhomotopic.

(This shows that $\pi_n(S^1) = 0$ for all $n \geq 2$, where π_n is the n-th homotopy group.)

Problem 3.C. (a) Let $p : (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ be a covering space such that X is path-connected. Prove that the following are equivalent.

(i) the subgroup $\operatorname{Im}(\pi_1(p)) \leq \pi_1(X, x_0)$ has index k;

(ii) $|p^{-1}(x_0)| = k$;

(iii) $|p^{-1}(x)| = k$ for all $x \in X$.

(b) Suppose that F_m is a subgroup of index k in F_n for some positive integers m, k, and n. Prove that $m - 1 = k(n - 1)$.

Problem 3.D. Let X be a topological space. Suppose that a group G has a continuous left action on X that is **properly discontinuous**, meaning that for every $g \neq 1 \in G$ and point $x \in X$, there exists an open neighborhood $x \in U$ such that $U \cap gU = \emptyset$.

(a) Let X/G be the quotient $X/\sim$, where $x \sim y$ if and only if $y = gx$ for some $g \in G$. Prove that X is a covering space of X/G.

(b) If X is a simply connected, prove that $\pi_1(X/G) \simeq G$.

Special cases of this result include $\pi_1(S^1) = \mathbb{Z}$ and $\pi_1(\mathbb{RP}^n) = \mathbb{Z}/2$.

Problem 3.E. Let G be a path-connected and locally path-connected topological group (a topological group is a group such that multiplcation map $m : G \times G \to G$ and inverse map $i : G \to G$ are both continuous). Let $p : \widetilde{G} \to G$ be a covering space of G such that $\widetilde{G}$ is path-connected. Let e be the identity of G, and let $\widetilde{e} \in p^{-1}(e)$ be a point in the fiber of e.

(a) Prove that there exists a unique group structure of $\widetilde{G}$ such that $\widetilde{e}$ is identity and p is a group homomorphism.

(b) Prove that $\operatorname{Ker} p$ is contained in the center of $\widetilde{G}$.

(Depending on your approach, you might use Problem 2.C or might get an alternative solution to Problem 2.C by taking $\widetilde{G}$ to be the universal cover of G.)

4 Homology

In Chapter 2, we have met our first algebraic invariant π_1 that was based on the loop.

It is natural to extend this to higher dimensional loop. Similar to π_1 is the higher **homotopy groups** $\pi_n(X)$, which is a group of maps $S^n \to X$ up to deformation. It turns out that $\pi_n(X)$ is abelian for all $n \geq 2$. Unfortunately, the groups $\pi_n(X)$ are very chaotic and very difficult to compute. For example, the first few homotopy groups of S^2 are shown below.

$\pi_1(S^2)$	$\pi_2(S^2)$	$\pi_3(S^2)$	$\pi_4(S^2)$	$\pi_5(S^2)$	$\pi_6(S^2)$
0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/12$

In this chapter, we will develop a simpler algebraic invariant that

- captures high-dimensional properties of a topological space, while
- is easy to compute.

This new invariant, **homology groups** will be based on the **number of holes** in a topological space.

Example 4.0.1.

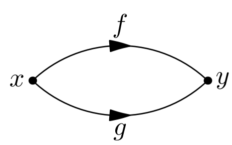
We have the following:

- Y has no holes.
- O has one hole.
- 8 has two holes.

We want to design an invariant that will capture these.

We are now going to explain how to compute H_1 without proving anything.

► Homology of letter "O". Let's compute a homology of a circle. First, we break up the circle to a directed graph.



We let $\mathbb{Z}[f, g]$ be a free abelian group generated by f and g. An element of $\mathbb{Z}[f, g]$ is a linear combination of f and g, such as $4f + 2g$, $3f - 2g$, or 0. Similarly, let $\mathbb{Z}[x, y]$ be the free abelian group on generators x and y.

We now define the homomorphism

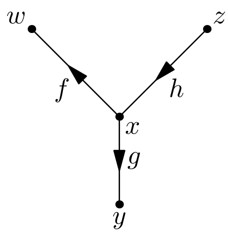
$$\partial : \mathbb{Z}[f, g] \to \mathbb{Z}[x, y]$$

by sending each edge to target minus source. In our case, we have $\partial(f) = y - x$ and $\partial(g) = y - x$. We can now define

$$H_1 = \ker \partial,$$

which turns out to be $\mathbb{Z}[f-g]$. Thus, $H_1(\mathsf{O}) = \mathbb{Z}$.

▶ Homology of letter "Y" Let's do another example, on Y. Let's make the following directed graph.



In this case, we have the map

$$\partial : \mathbb{Z}[f,g,h] \to \mathbb{Z}[x,y,z,w]$$

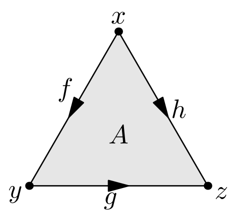
defined by

$$\begin{aligned} \partial(f) &= w - x \\ \partial(g) &= y - x \\ \partial(h) &= x - z, \end{aligned}$$

and one can check that $H_1 = \ker(\partial) = 0$.

So far, we have computed only homology of one-dimensional shapes. We now want to do two-dimensional shapes.

▶ Homology of a solid triangle. Let's take a solid triangle, for example.



Let A denote the triangle with edges f, g, h. As usual, we have the boundary map

$$\partial_1 : \mathbb{Z}[f,g,h] \to \mathbb{Z}[x,y,z]$$

defined by

$$\begin{aligned} \partial_1(f) &= y - x \\ \partial_1(g) &= z - y \\ \partial_1(h) &= z - x. \end{aligned}$$

However, this doesn't capture the fact that the triangle is solid. To do that, we have another boundary map

$$\partial_2 : \mathbb{Z}[A] \to \mathbb{Z}[f,g,h],$$

given by summing along the edges of the triangle. In this case, $\partial_2(A) = f + g - h$.

Then, the homology is

$$H_1(X) = \frac{\operatorname{Ker} \partial_1}{\operatorname{Im} \partial_2} = \frac{\mathbb{Z}[f+g-h]}{\mathbb{Z}[f+g-h]} = 0.$$

In general, the homology group will be kernel of one map, mod image of another map.

For this chapter, we will explain how to define this rigorously. In particular, we will

- explain what it means to triangulate a topological space;
- explain how to compute for triangulated space, $H_1(X)$, $H_2(X)$, $\ldots$, where $H_n(X)$ will count n-dimensional holes.
- explain why $H_n(X)$ depends only on the topological space itself.

Unfortuntely, the definition of homology is very technical, and it takes most of the chapter to prove enough properties that we can compute homology of simple spaces.

§4.1 Definition of Singular Homology

§4.1.1 Semisimplicial Sets

First, we need a model for edges, triangles, or higher-dimensional constructs.

Definition 4.1.1 (Semisimplicial Sets).

A **semisimplicial set** X_* is a sequence of sets $X_0, X_1, X_2, \ldots$ and functions

$$\begin{aligned} d_0, d_1 &: X_1 \to X_0 \\ d_0, d_1, d_2 &: X_2 \to X_1 \\ d_0, d_1, d_2, d_3 &: X_3 \to X_2, \quad \text{etc.} \end{aligned}$$

that satisfies **simplicial identities**

$$d_i \circ d_j = d_{j-1} \circ d_i$$

for all $i < j$ such that both sides make sense.

Example 4.1.2 (Semisimplicial set corresponding to circle).

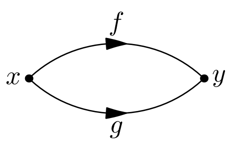
Suppose that X_* is a semisimplicial set such that

$$X_2 = \emptyset, X_3 = \emptyset, X_4 = \emptyset, \ldots,$$

so we have only two functions $d_0, d_1 : X_1 \to X_0$, so simplicial identities holds.

One should think of this as a directed graph, where $X_1 = \{\text{edges}\}$, $X_0 = \{\text{vertices}\}$, d_0 is the target map, and d_1 is the source map.

For example, consider the circle example that we did in the previous section:



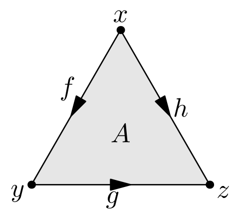
We have $X_1 = \{f, g\}$, $X_0 = \{x, y\}$, and we have

$$d_1(f) = d_0(f) = x, \qquad d_1(g) = d_0(g) = y.$$

Now, let's look at something for which X_2 is not empty, let's say a solid triangle in the above example.

Example 4.1.3 (Semisimplicial set corresponding to solid triangle)**.**

Consider a solid triangle in the previous section.



This corresponds to a semisimplicial set

$$X_0 = \{x, y, z\}, \qquad X_1 = \{f, g, h\}, \qquad X_2 = \{A\}.$$

Then, as usual, we have the map with vertices and edges:

$$\begin{aligned} d_0 f &= y \qquad & d_0 g &= z \qquad & d_0 h &= z \\ d_1 f &= x \qquad & d_1 g &= y \qquad & d_1 h &= x. \end{aligned}$$

For map in X_2, we have

$$d_0 A = g, \qquad d_1 A = h, \qquad d_2 A = f.$$

This is only one way to assign the map. Note that it satisfies simplicial identities:

$$d_0 d_2 A = d_0 f = y, \qquad d_1 d_0 A = d_1 g = y.$$

§4.1.2 Homology of Semisimplicial Set

We now define a homology group, which generalizes previous section's discussion.

Definition 4.1.4 (Chain complex)**.**

A **chain complex** is a sequence

$$\cdots \xrightarrow{\partial_3} A_2 \xrightarrow{\partial_2} A_1 \xrightarrow{\partial_1} A_0 \xrightarrow{\partial_0} A_{-1} \xrightarrow{\partial_{-1}} A_{-2} \xrightarrow{\partial_{-2}} \cdots$$

such that $\partial_{i-1} \circ \partial_i = 0$ for all i (i.e., $\operatorname{Im} \partial_i \subseteq \operatorname{Ker} \partial_{i-1}$).

A morphism f from (A_*, ∂_*) to (B_*, ∂_*) is a diagram

$$\begin{tikzcd} \cdots \arrow[r] & A_2 \arrow[r, "\partial_2"] \arrow[d, "f_2"] & A_1 \arrow[r, "\partial_1"] \arrow[d, "f_1"] & A_0 \arrow[r, "\partial_0"] \arrow[d, "f_0"] & A_{-1} \arrow[r, "\partial_{-1}"] \arrow[d, "f_{-1}"] & \cdots \\ \cdots \arrow[r] & B_2 \arrow[r, "\partial_2"] & B_1 \arrow[r, "\partial_1"] & B_0 \arrow[r, "\partial_0"] & B_{-1} \arrow[r, "\partial_{-1}"] & \cdots \end{tikzcd}$$

such that all squares commute. We call such morphism a **chain map**.

Chain complexes form a category **chAb**.

Definition 4.1.5 (S_n)**.**

Given a semisimplicial set X_*, we let

$$S_n(X_*) = \mathbb{Z}[X_n] = \text{free abelian group on } X_n.$$

We then define the **boundary map**

$$\partial_n : S_n(X_*) \to S_{n-1}(X_*)$$
$$\partial_n(\sigma) \mapsto \sum_{k=0}^{n} (-1)^k d_k(\sigma).$$

In the boundary case, we make the convention that $S_{-1}(X_*) = \{0\}$ and ∂_0 is the zero homomorphism.

The set $S_n(X_*)$, together with the boundary map ∂_n forms the **singular chain complex**

$$\dots \longrightarrow S_{n+1}(X_*) \xrightarrow{\partial_{n+1}} S_n(X_*) \xrightarrow{\partial_n} S_{n-1}(X_*) \longrightarrow \dots .$$

We generally omit subscript and write ∂ instead of ∂_n.

Example 4.1.6.

In the triangle example Example 4.1.3, we have

$$\partial_2(A) = d_0 A - d_1 A + d_2 A = g - h + f = f + g - h.$$

In order to check that the above is a chain complex, we have to check the following exercise. The proof is just grinding out definitions.

Exercise 4.1.7. Using the definition of ∂, show that $\partial_n \circ \partial_{n+1} = 0$ for any semisimplicial set X.

We now define a homology group for a semisimplicial set.

Definition 4.1.8 (Cycle, boundaries, and homology groups)**.**

For a semisimplicial set X_*, define

$$\begin{aligned} \text{the group of } n\text{-}\textbf{cycles} \quad & Z_n(X_*) = \mathrm{Ker}(\partial_n), \\ \text{the group of } n\text{-}\textbf{boundaries} \quad & B_n(X_*) = \mathrm{Im}(\partial_{n+1}), \\ \text{the } n\text{-th homology group} \quad & H_n(X_*) = Z_n(X_*)/B_n(X_*). \end{aligned}$$

(This can be defined more generally on any chain complex, since we only care about the kernels and images of the boundary maps.)

Note that $B_n(X_*) \subseteq Z_n(X_*)$ always. This is equivalent to $\partial_n \circ \partial_{n+1} = 0$,

§4.1.3 Homology of Topological Space

To compute homology of a topological space, one needs to simply cut the space into simplices, setting up the corresponding semisimplicial sets. However, it is very non-obvious fact that homology group is independent of how we cut it.

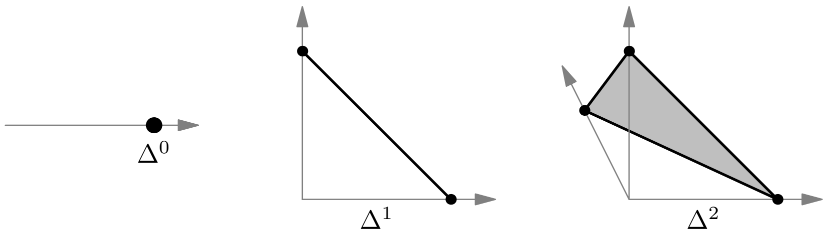
Thus, we will side-step this and consider the *canonical* semisimplicial set of any topological space X. After we develop the tools to understand this, we will later go back to actuaully cutting X.

Definition 4.1.9 (Standard n-simplex)**.**

For any $n \geq 0$, the **standard n-simplex** Δ^n is a subspace of $\mathbb{R}^{n+1}$. It is the convex hull of the standard basis vectors:

$$\Delta^n = \left\{(t_0, t_1, \ldots, t_n) \in \mathbb{R}^{n+1} \;:\; t_i \geq 0 \text{ for all } i \text{ and }, t_0 + t_1 + \cdots + t_n = 1\right\}.$$

For example, here are Δ^0, Δ^1, and Δ^2 embedded in the corresponding coordinate space.

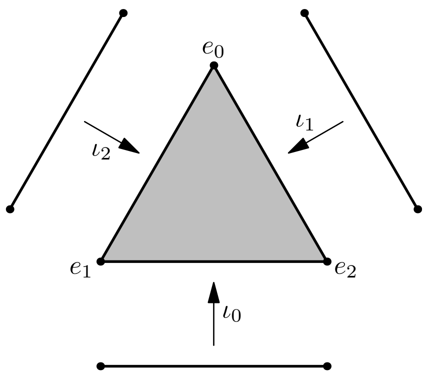


In particular, Δ^0 is a point, Δ^1 is a line segment, Δ^2 is a triangle, Δ^3 is a tetrahdron, and so on.

Definition 4.1.10 (Face Inclusion)**.**

For every i and n, the i-th **face inclusion** $\iota_i : \Delta^{n-1} \to \Delta^n$ is the map that sends $(t_0, t_1, \ldots, t_{n-1})$ to $(t_0, t_1, \ldots, t_{i-1}, 0, t_i, \ldots, t_{n-1})$.

For example, if $e_0 = (1, 0, 0)$, $e_1 = (0, 1, 0)$, and $e_2 = (0, 0, 1)$ are the standard basis, here are the three face-inclusions $\Delta^1 \to \Delta^2$.



Definition 4.1.11 (Semisimplicial set of a topological space)**.**

Given a topological space X and integer $n \geq 0$, we define

$$\mathrm{Sing}_n(X) = \{\text{continuous functions } \Delta^n \to X\}.$$

An element in $\mathrm{Sing}_n(X)$ is typically called an ***n*-chain**.

To promote $\mathrm{Sing}_*(X)$ to a semisimplicial set, we need to define the maps d_i. In particular, given any map $\sigma : \Delta^n \to X$, we compose it with the face inclusion ι_i to get a map $\sigma \circ \iota_i : \Delta^{n-1} \to X$. We define $d_i(\sigma) = \sigma \circ \iota$.

This gives rise to semisimplicial sets $\mathrm{Sing}_*(X)$. These sets will probably be uncountably infinite.

Many times when dealing with simplices, we typically indicate face of a simplex by the vertices alone. For example, suppose that $\sigma : \Delta^2 \to X$ is a 2-chain. Let e_0, e_1, e_2 is the standard basis of $\mathbb{R}^3$, which is the vertices of the standard Δ^2. Let $x_i = \sigma(e_i)$ for all $i = 0, 1, 2$. Then σ is a 2-simplex with vertices $[x_0, x_1, x_2]$.

Then $d_0(\sigma) = [x_1, x_2]$, which is a 1-chain with vertices x_1 and x_2. When writing this, we ignore the information of where points in between the segment goes to, but it is understood that it comes from σ and the standard face inclusion.

Similarly, we write

$$d_0(\sigma) = [x_1, x_2], \quad d_1(\sigma) = [x_0, x_2], \quad d_2(\sigma) = [x_0, x_1]$$

$$\partial\sigma = [x_1, x_2] - [x_0, x_2] + [x_0, x_1].$$

As another example, if $\sigma : \Delta^3 \to X$ is a 3-chain with vertices x_0, x_1, x_2, x_3. Then

$$\partial\sigma = [x_1, x_2, x_3] - [x_0, x_2, x_3] + [x_0, x_1, x_3] - [x_0, x_1, x_2].$$

We now finally define the homology group of a topological space.

Definition 4.1.12 (Singular Homology)**.**

If X is a topological space, we define

$$\begin{aligned} S_n(X) &= S_n(\mathrm{Sing}_*(X)) \\ Z_n(X) &= Z_n(\mathrm{Sing}_*(X)) \\ B_n(X) &= B_n(\mathrm{Sing}_*(X)) \\ H_n(X) &= H_n(\mathrm{Sing}_*(X)). \end{aligned}$$

We now have the definition of homology groups, except that these groups are generated by uncountably infinitely many elements, so we don't really know how to compute them. For the rest of the chapter, we will develop tools to get a handle on these groups and eventually be able to compute them.

§4.1.4 Everything is a Functor

We note that H_n can be promoted to a functor $\mathbf{Top} \to \mathbf{Ab}$. To show this, we have to show that every step of the construction is a functor.

- Note that Sing_n is a functor $\mathbf{Top} \to \mathbf{Sets}$ is a functor. To see this, recall that $\mathrm{Sing}_n(X)$ is the set of continuous functions $\Delta^n \to X$. To get a functor, we need to describe what Sing_n does to a morphism. If $f : X \to Y$ is a morphism of $\mathbf{Top}$, then any element in $\mathrm{Sing}_n(X)$ (a map from $\Delta^n \to X$) compose to $\Delta^n \to X \xrightarrow{f} Y$, obtaining a map $\Delta^n \to Y$, which is in $\mathrm{Sing}_n(Y)$. This gives a morphism $\mathrm{Sing}_n(X) \to \mathrm{Sing}_n(Y)$.
- The map that sends a set X to a free abelian group generated by X is a functor $\mathbf{Sets} \to \mathbf{Ab}$. See if you can guess the definition of how it does on the map on this one. Composing this with Sing_n promotes S_n to a functor.
- For any continuous map $f : X \to Y$, note that $d_i : \mathrm{Sing}_n(X) \to \mathrm{Sing}_{n-1}(X)$ is a natural transformation. In particular, the diagram then the diagram

$$\begin{tikzcd} \mathrm{Sing}_n(X) \arrow[r, "d_i"] \arrow[d, "\mathrm{Sing}_n(f)"] & \mathrm{Sing}_{n-1}(X) \arrow[d, "\mathrm{Sing}_{n-1}(f)"] \\ \mathrm{Sing}_n(Y) \arrow[r, "d_i"] & \mathrm{Sing}_{n-1}(Y) \end{tikzcd}$$

  commutes. This means that $d_i \circ \mathrm{Sing}_n(f) = \mathrm{Sing}_{n-1}(f) \circ d_i$.

  By summing various d_i, this implies that $\partial : S_n(X) \to S_{n-1}(X)$ is also a natural transformation. In other words, the diagram

$$\begin{tikzcd} \dots \arrow[r] & S_{n+1}(X) \arrow[r, "\partial"] \arrow[d, "S_{n+1}(f)"] & S_n(X) \arrow[r, "\partial"] \arrow[d, "S_n(f)"] & S_{n-1}(X) \arrow[r, "\partial"] \arrow[d, "S_{n-1}(f)"] & \dots \\ \dots \arrow[r] & S_{n+1}(Y) \arrow[r, "\partial"] & S_n(Y) \arrow[r, "\partial"] & S_{n-1}(Y) \arrow[r, "\partial"] & \dots \end{tikzcd}$$

commutes. This means that the map $S_n(f) : S_n(X) \to S_n(Y)$ sends an element in $Z_n(X)$ to an element in $Z_n(Y)$ and sends an element in $B_n(X)$ to an element in $B_n(Y)$. Thus, it defines a map

$$Z_n(f) : Z_n(X) \to Z_n(Y), \quad B_n(f) : B_n(X) \to B_n(Y), \quad H_n(f) : H_n(X) \to H_n(Y)$$

promoting Z_n, B_n, and H_n to a functor.

Thus, H_n is a functor.

§4.2 Homotopy Invariance

Armed with language of category, we will now work up to computing homology groups of interesting spaces. By the end of this section, we will have shown that for any convex set $X \subset \mathbb{R}^k$, we have

$$H_n(X) = \begin{cases} \mathbb{Z} & n = 0 \\ 0 & n > 0 \end{cases}.$$

§4.2.1 Homology Group of a Point

We will start with computing a homology group of a point. Let $X = \{p\}$ be a point. Then, we have

- $\mathrm{Sing}_n(X) = \{a_n\}$, where a_n is the unique continuous function sending the entire Δ^n to p.
- The semisimplicial set is evident: there is one option, which is to have all of $d_0, d_1, \dots, d_n$ maps $a_n \in \mathrm{Sing}_n(X)$ to $a_{n-1} \in \mathrm{Sing}_{n-1}(X)$.
- We now form a chain complex S_*. Since there is only one thing in $\mathrm{Sing}_n(X)$, the chain complex looks like
$$\cdots \to \mathbb{Z} \xrightarrow{\partial_2} \mathbb{Z} \xrightarrow{\partial_1} \mathbb{Z} \xrightarrow{\partial_0} 0 \longrightarrow 0 \longrightarrow \dots.$$
- Next, we compute boundary maps: in small cases, we have
$$\begin{aligned} \partial_0(a_0) &= 0 \\ \partial_1(a_1) &= d_0 a_1 - d_1 a_1 = 0 \\ \partial_2(a_2) &= d_0 a_2 - d_1 a_2 + d_2 a_2 = a_1 \\ \partial_3(a_3) &= 0 \\ \partial_4(a_4) &= a_3 \\ &\vdots \end{aligned}$$
- Finally, we compute the homology:
$$\begin{aligned} H_0(X) &= \mathrm{Ker}(\partial_0)/\operatorname{Im}(\partial_1) = \mathbb{Z}/0 = \mathbb{Z} \\ H_1(X) &= \mathrm{Ker}(\partial_1)/\operatorname{Im}(\partial_2) = \mathbb{Z}/\mathbb{Z} = 0 \\ H_2(X) &= \mathrm{Ker}(\partial_2)/\operatorname{Im}(\partial_3) = 0/0 = 0 \\ H_3(X) &= \mathrm{Ker}(\partial_3)/\operatorname{Im}(\partial_4) = \mathbb{Z}/\mathbb{Z} = 0, \\ &\vdots \end{aligned}$$

In conclusion, we have that

Proposition 4.2.1.

The homology of a point is given by

$$H_n(\{p\}) = \begin{cases} \mathbb{Z} & n = 0 \\ 0 & n \neq 0 \end{cases}.$$

Unfortunately, this is about as far as we can go using just definitions.

The following are two more properties of homology groups that we can quickly show now.

Exercise 4.2.2 (Zeroth Homology). Let X be a topological space. Prove that $H_0(X)$ is a free abelian group generated by path components of X.

Exercise 4.2.3 (Homology of disjoint union). Let X and Y be topological spaces. Prove that $H_n(X \amalg Y) = H_n(X) \oplus H_n(Y)$, where $X \amalg Y$ is the disjoint union of X and Y.

§4.2.2 Chain Homotopy

In order to prove homotopy invariance, we first define, completely algebraically, what it means for two chain complexes to be homotopic. This leads to the notion of **chain homotopy**.

Definition 4.2.4 (Chain homotopy).

Let C_* and D_* be chain complexes and $f_0, f_1 : C_* \to D_*$ in **chAb**. A **chain homotopy** $h : f_0 \simeq f_1$ is a collection of homomorphisms

$$h : C_n \to D_{n+1}$$

such that

$$\partial h + h\partial = f_1 - f_0.$$

A chain homotopy is a collection of maps in the following **non-commuting** diagram

$$\begin{tikzcd}
\dots & \dots \arrow[r, "\partial"] & C_{n+1} \arrow[r, "\partial"] \arrow[dl, "h"'] & C_n \arrow[r, "\partial"] \arrow[dl, "h"'] & C_{n-1} \arrow[r, "\partial"] \arrow[dl, "h"'] & \dots \\
\dots \arrow[r, "\partial"] & D_{n+2} \arrow[r, "\partial"] & D_{n+1} \arrow[r, "\partial"] & D_n \arrow[r, "\partial"] & \dots & \dots
\end{tikzcd}$$

that satisfies the chain homotopy condition stated above.

Proposition 4.2.5 (Chain homotopy induces isomorphism on homology groups).

If $f_0, f_1 : C_* \to D_*$ are chain homotopic, then $H_n(f_0) = H_n(f_1)$ as homomorphisms from $H_n(C_*) \to H_n(D_*)$.

► *Proof.* Suppose that $z \in Z_n(C_*)$ is an n-cycle. We must show that $f_1(z) - f_0(z) \in B_n(D_*)$. To do that, we use the chain homotopic conditin.

$$\begin{aligned} f_1(z) - f_0(z) &= \partial h(z) - h\partial(z) \\ &= \partial h(z) + h(0) = \partial h(z) \in B_n(D_*). \end{aligned}$$ □

§4.2.3 Homology of Star-shaped Sets

We now prove our first nontrivial result, that homology of star-shaped set is equal to that of a point. First, let us define star-shaped sets precisely.

Definition 4.2.6 (Star-shaped sets)**.**

A subspace $X \subseteq \mathbb{R}^n$ is **star-shaped** with respect to point $b \in X$ if, for all $x \in X$, the segment between x and b,

$$\{tx + (1-t)b : t \in [0,1]\}$$

is entirely within X.

Theorem 4.2.7.

If $X \subseteq \mathbb{R}^n$ is star-shaped with respect to a point $b \in X$, then

$$H_n(X) = \begin{cases} \mathbb{Z} & n = 0 \\ 0 & n \neq 0. \end{cases}$$

▶ *Proof of Theorem 4.2.7.* Suppose that X is star-shaped with respect to $b \in X$. We let C_* be the chain complex with

$$C_n = \begin{cases} \mathbb{Z} & n = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Then, it's easy to see that

$$H_n(C_*) = \begin{cases} \mathbb{Z} & n = 0 \\ 0 & \text{otherwise.} \end{cases}$$

We will show that C_* and $S_*(X)$ have the same homology groups. To do that, we define morphisms in both ways.

$$\begin{aligned} \varepsilon : S_*(X) &\to C_* \\ (x \in S_0(X)) &\mapsto 1 \end{aligned}, \qquad \begin{aligned} \eta : C_* &\to S_*(X) \\ (1 \in C_0) &\mapsto (b \in S_0(X)) \end{aligned}$$

Both morphisms commute with the boundary map (convince yourself that there is nothing to check here).

We claim that $H_n(\varepsilon)$ and $H_n(\eta)$ are inverse isomorphisms of abelian groups for all n. In one direction, note that $\varepsilon \circ \eta$ is an identity map. However, $\eta \circ \varepsilon$ is far from identity. In particular, we have that

$$\eta(\varepsilon(\sigma)) = \begin{cases} b & \text{if } \sigma \in S_0(X) \\ 0 & \text{otherwise.} \end{cases}$$

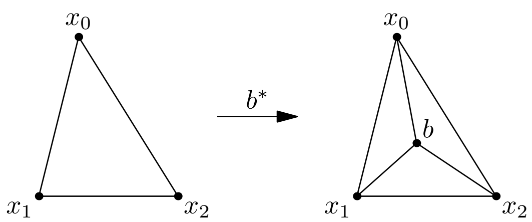
We claim that $\eta \circ \varepsilon$ and $\mathrm{id}_{S_*(X)}$ are chain homotopic. To do this, we define the chain homotopy

Define $b^* : S_n(X) \to S_{n+1}(X)$ that takes

$$\sigma : \Delta^q \to X \quad \text{to} \quad b^*(\sigma) : \Delta^{n+1} \to X \text{ defined by}$$

$$b^*(\sigma)(t_0, t_1, \ldots, t_{n+1}) = \begin{cases} b & t_0 = 1 \\ t_0 b + (1-t_0)\sigma\left(\frac{t_1}{1-t_0}, \ldots, \frac{t_{n+1}}{1-t_0}\right) & t_0 \neq 1. \end{cases}$$

For example, if $n = 1$, then the map takes segment x_0x_1 (as a map $\Delta^1 \to X$) to the triangle bx_0x_1 (as a map $\Delta^2 \to X$). In general, the map takes an n-simplex to an $(n+1)$-simplex with an extra vertex b.



Observe that $d_0(b^*(\sigma)) = \sigma$ and $d_i(b^*(\sigma)) = b^*(d_{i-1}\sigma)$. We now check the chain homotopy criterion by spliting into two cases.

- **If $n \geq 1$**, then
$$\begin{aligned}\partial b_n^*(\sigma) &= \sum_{k=0}^{n} (-1)^k d_k b_n^* \sigma \\ &= \sigma + \sum_{k=1}^{n} (-1)^k b_{n-1}^*(d_{k-1}\sigma) \\ &= \sigma - b_{n-1}^* \left(\sum_{k=0}^{n} (-1)^k d_k \sigma \right) = \sigma - b_{n-1}^*(\partial\sigma),\end{aligned}$$
so b_q^* indeed satisfies the chain homotopy criterion.
- **If $n = 0$**, then let σ be point x. We compute $\partial b_0^*(x) = x - b$, $b_0^*(\partial x) = 0$, and $\eta(\varepsilon(x)) = b$. Thus, we indeed have that $\partial b_0^*(x) + b_0^*(\partial x) = x - \eta(\varepsilon(x))$. □

§4.2.4 Homotopy Invariance

By the end of this subsection, we will prove the following.

Theorem 4.2.8 (Homotopy Invariance)**.**

If f and g are homotopic, then $S_*(f)$ and $S_*(g)$ are chain homotopic. Thus, $H_n(f) = H_n(g)$.

Corollary 4.2.9.

If X and Y are two homotopy-equivalent spaces, then $H_n(X) \simeq H_n(Y)$ for all n.

▶ *Proof.* Combine Theorem 4.2.8 with Proposition 4.2.5. □

We now prove Theorem 4.2.8.

The idea of the proof is as follows. Suppose that f and g are homotopic with homotopy $h : X \times [0,1] \to Y$. This induces the map
$$S_{n+1}(h) : S_{n+1}(X \times [0,1]) \to S_{n+1}(Y).$$
However, we need to define a chain homotopy $S_n(X) \to S_{n+1}(Y)$. Hence, we seek the natural transformation
$$k_X : S_n(X) \to S_{n+1}(X \times [0,1])$$
so that we can compose with $S_{n+1}(h)$ and obtain the desired chain homotopy. For each $\sigma : \Delta^n \to X$, naturality of k_X gives the commuting diagram
$$\begin{tikzcd} S_n(\Delta^n) \arrow[r, "S_n(\sigma)"] \arrow[d, "k_Y"] & S_n(X) \arrow[d, "k_X"] \\ S_{n+1}(\Delta^n \times [0,1]) \arrow[r, "S_{n+1}(\sigma \times \mathrm{id})"] & S_{n+1}(X \times [0,1]) \end{tikzcd}.$$

If we start from id_{Δ^n} on the top-left corner and trace through, we get the diagram

$$\begin{tikzcd}
\mathrm{id}_{\Delta^n} \arrow[r, mapsto, "S_n(\sigma)"] \arrow[d, mapsto, "k_{\Delta^n}"] & \sigma \arrow[d, mapsto, "k_X"] \\
k_{\Delta^n}(\mathrm{id}_{\Delta^n}) & k_X(\sigma)
\end{tikzcd}.$$

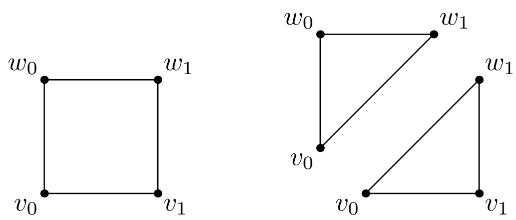
In particular, **$k_X(\sigma)$ is uniquely determined by $k_{\Delta^n}(\mathrm{id}_{\Delta^n})$**, by making it undergo through map $S_{n+1}(\sigma \times \mathrm{id})$. Thus, do define the natural transformation, we only need to specify where $k_{\Delta^n}(\mathrm{id}_{\Delta^n})$ goes to in $S_{n+1}(\Delta^n \times [0,1])$. Intuitively, we will be cutting $\Delta^n \times [0,1]$ into several $(n+1)$-simplices.

Let $u_0, \dots, u_n$ be the vertices of Δ^n. For each $i = 0, 1, \dots, n$, let $v_i = (u_i, 0) \in \Delta^n \times [0,1]$ and $w_i = (u_i, 1) \in \Delta^n \times [0,1]$. Let's work out some small examples.

- **If $n = 1$,** then we have to cut the square with vertices v_0, v_1, w_0, w_1 into two triangles. For the computation to work out, we use

$$k_{\Delta^1}(\mathrm{id}_{\Delta^1}) = [v_0, v_1, w_1] - [v_0, w_0, w_1],$$

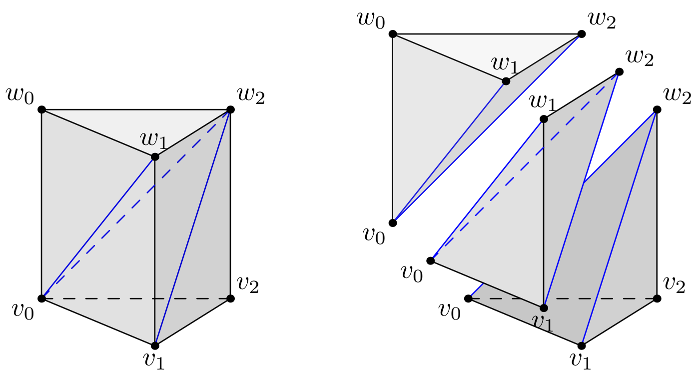
  corresponding to the cut below:



- **If $n = 2$,** then we are cutting a triangular prism to three tetrahedra. We choose

$$k_{\Delta^2}(\mathrm{id}_{\Delta^2}) = [v_0, v_1, v_2, w_2] - [v_0, v_1, w_1, w_2] + [v_0, w_0, w_1, w_2],$$

  corresponding to the below diagram



One can turn this to a general construction, and you might have guessed what the pattern is already. This is the proof used in [Hat02], which the reader can read. Below, we will give an alternative proof that utilizes Theorem 4.2.7 and relies less on explicit simplex calculation. In particular, we will use Theorem 4.2.7 to say that since $\Delta^n \times [0,1]$ is star-shaped, it can be cut into simplicies without specifying how exactly to cut it.

▶ *Proof.* Define maps

$$\begin{aligned} \iota_{X,0} : X &\to X \times [0,1] \\ x &\mapsto (x, 0) \end{aligned} \quad \text{and} \quad \begin{aligned} \iota_{X,1} : X &\to X \times [0,1] \\ x &\mapsto (x, 1) \end{aligned}.$$

We claim that we can define k_X such that

$$\partial k_X + k_X \partial = S_n(\iota_{X,1}) - S_n(\iota_{X,0}). \tag{4.1}$$

To do this, we use induction on n to define $k_X : S_n(X) \to S_{n+1}(X \times [0,1])$. The base case $n = 0$ is clear by sending $k_X(p)$ to the 1-simplex joining $(p, 0)$ and $(p, 1)$ for each point $p \in X$.

To get the inductive step, we first define $k_{\Delta^n}(\mathrm{id}_{\Delta^n})$. We need that to satisfy

$$\partial k_{\Delta^n} = -k_{\Delta^n}\partial(\mathrm{id}_{\Delta^n}) + S_n(\iota_{X,1})(\mathrm{id}_{\Delta^n}) - S_n(\iota_{X,0})(\mathrm{id}_{\Delta^n}). \tag{4.2}$$

We note that applying ∂ to the right hand side gives

$$\begin{aligned}
&-\partial(k_{\Delta^n}\partial(\mathrm{id}_{\Delta^n})) + \partial(S_n(\iota_{\Delta^n,1})(\mathrm{id}_{\Delta^n})) - \partial(S_n(\iota_{\Delta^n,0})(\mathrm{id}_{\Delta^n})) \\
&= -\partial(k_{\Delta^n}\partial(\mathrm{id}_{\Delta^n})) + S_n(\iota_{\Delta^n,1})(\partial(\mathrm{id}_{\Delta^n})) - S_n(\iota_{\Delta^n,0})(\partial(\mathrm{id}_{\Delta^n})) \\
&= k_X\partial(\partial(\mathrm{id}_{\Delta^n})) = 0
\end{aligned}$$

where we use naturality of ∂ to commute with S_n, the inductive hypothesis, and the fact that $\partial^2 = 0$ in each line, respectively. Thus, the right hand side is a cycle in $S_n(\Delta^n \times [0,1])$. However, $H_n(\Delta^n \times [0,1]) = 0$ by Theorem 4.2.7, so any cycle is a boundary! In particular, the right hand side is an image of Δ, and we can just take k_{Δ^n} to be any preimage! We thus make (4.2) satisfied.

We can now check this identity for any X. For any $\sigma : \Delta^n \to X$, we have by naturality that

$$\partial k_X(\sigma) + k_X(\partial(\sigma)) = S_{n+1}(\sigma \times \mathrm{id})(\partial k_{\Delta^n}(\mathrm{id}_{\Delta^n}) + k_{\Delta^n}\partial(\mathrm{id}_{\Delta^n})),$$

and similarly,

$$S_n(\iota_{X,1})(\sigma) - S_n(\iota_{X,0})(\sigma) = S_n(\sigma \times \mathrm{id})(\iota_{\Delta^n,0}) - S_n(\sigma \times \mathrm{id})(\iota_{\Delta^n,0}).$$

Since (4.2) is satisfied, we get that (4.1) is satisfied. This completes the induction.

Finally, our chain homotopy is $H = S_n(h) \circ k_X$. We have by naturality of ∂ that

$$\begin{aligned}
\partial H + H\partial &= S_n(h)(\partial k_X + k_X\partial) \\
&= S_n(h)(S_n(\iota_{X,1}) - S_n(\iota_{X,0})) && \text{(using (4.1))} \\
&= S_n(h \circ \iota_{X,1}) - S_n(h \circ \iota_{X,0}) \\
&= S_n(g) - S_n(f),
\end{aligned}$$

implying that H is a valid chain homotopy. □

§4.3 Exact Sequences

Before exploring more about homology, we take an algebra detour and learn about exact sequence, which will be a tool to express various results involving maps.

§4.3.1 Exact Sequences of Abelian Group

Definition 4.3.1 (Exact sequence of abelian groups).

A sequence

$$A \xrightarrow{f} B \xrightarrow{g} C$$

of abelian groups is **exact** if $\operatorname{Ker} g = \operatorname{Im} f$.

A longer sequence is exact if each three adjacent terms are exact.

From the definition, one can see that

- $0 \longrightarrow A \xrightarrow{f} B$ is exact iff f is injective.
- $A \xrightarrow{f} B \to 0$ is exact iff f is surjective.
- $0 \longrightarrow A \xrightarrow{f} B \longrightarrow 0$ is exact iff f is an isomorphism.

Example 4.3.2 (The "mod 2" exact sequence).

The diagram

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

is exact.

Example 4.3.3 (Chain complex is usually not exact).

$S_*(X)$ is exact if and only if all homology groups vanish, so it's usually not exact. In fact, the homology groups measure how far a chain complex is from being exact.

We have the following basic properties of exact sequence that is very useful.

Theorem 4.3.4 (Five Lemma).

Consider a diagram

$$\begin{tikzcd}
A_4 \arrow[r, "\partial"] \arrow[d, "f_4"] & A_3 \arrow[r, "\partial"] \arrow[d, "f_3"] & A_2 \arrow[r, "\partial"] \arrow[d, "f_2"] & A_1 \arrow[r, "\partial"] \arrow[d, "f_1"] & A_0 \arrow[d, "f_0"] \\
B_4 \arrow[r, "\partial"] & B_3 \arrow[r, "\partial"] & B_2 \arrow[r, "\partial"] & B_1 \arrow[r, "\partial"] & B_0
\end{tikzcd}$$

such that all squares commute and both rows are exact. If f_4, f_3, f_1, f_0 are isomorphisms, then so is f_2.

▶ *Proof.* Let us prove that f_2 is surjective. The proof that f_2 is injective will be left as an exercise.

Suppose $b_2 \in B_2$. Thus, $\partial b_2 = f_1(a_1)$ for some a_1. However, note that

$$f_0(\partial a_1) = \partial f_1(a_1) = \partial\partial b_2 = 0,$$

so $\partial a_1 = 0$. Thus, by exactness at A_1, there exists $a_2 \in A_2$ such that $\partial a_2 = a_1$. The steps up to here can be summarized at the diagram below.

$$\begin{tikzcd}
a_2 \arrow[r, mapsto] & a_1 \arrow[r, mapsto] \arrow[d, mapsto] & 0 \arrow[d, mapsto] \\
b_2 \arrow[r, mapsto] & \partial b_2 \arrow[r, mapsto] & 0
\end{tikzcd}$$

It would be nice if $b_2 = f_2(a_2)$, but this is not true, so we need to find the *correction term*. We need to now use the left diagram. Next, we note that

$$\partial(b_2 - f_2(a_2)) = \partial b_2 - \partial f_2(a_2) = \partial b_2 - f_1(\partial a_2) = \partial b_2 - \partial b_2 = 0.$$

Thus, by exactness at B_2, there exists $b_3 \in B_3$ such that $\partial b_3 = b_2 - f_2(a_2)$. Then, we can find a_3 such that $f_3(a_3) = b_3$.

Finally, we have

$$f_2(a_2 + \partial a_3) = f_2(a_2) + f_2(\partial a_3) = f_2(a_2) + \partial(f_3(a_3)) = f_2(a_2) + \partial b_3 = b_2,$$

as desired. □

Exercise 4.3.5. Complete the proof of Five lemma by showing that f_2 is injective.

Remark 4.3.6.

A stronger statement of Five lemma is

- If f_1 and f_3 are surjective and f_0 is injective, then f_2 is surjective.
- If f_1 and f_3 are injective and f_4 is surjective, then f_2 is injective.

The first bullet point is proved by above.

The most common class of exact sequence is a short exact sequence, defined below.

Definition 4.3.7 (Short exact sequence).

A **short exact sequence** of abelian groups is an exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0.$$

Here is a quick exercise to help you digest what this means.

Exercise 4.3.8. Prove that if $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ is an exact sequence, then

(a) $A \simeq \operatorname{Ker} g$.

(b) $C \simeq B / \operatorname{Im} f$.

Example 4.3.9 (Ring morphism induces short exact sequence).

For any morphism $f : A \to B$, we have the short exact sequence

$$0 \longrightarrow \operatorname{Ker} f \longrightarrow A \longrightarrow \operatorname{Im} f \longrightarrow 0.$$

§4.3.2 Exact Sequence of Chain Complexes and Snake Lemma

Definition 4.3.10 (Short exact sequence of chain complexes).

A **short exact sequence of chain complexes** is a sequence in **chAb**

$$0_* \longrightarrow A_* \longrightarrow B_* \longrightarrow C_* \longrightarrow 0_*$$

such that for each integer n,

$$0 \longrightarrow A_n \longrightarrow B_n \longrightarrow C_n \longrightarrow 0$$

is exact.

A short exact sequence of chain complexes is defined by defining all the vertical arrows so that they commute with ∂.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\dots \arrow[r] & A_{n+1} \arrow[r, "\partial"] \arrow[d] & A_n \arrow[r, "\partial"] \arrow[d] & A_{n-1} \arrow[r] \arrow[d] & \dots \\
\dots \arrow[r] & B_{n+1} \arrow[r, "\partial"] \arrow[d] & B_n \arrow[r, "\partial"] \arrow[d] & B_{n-1} \arrow[r] \arrow[d] & \dots \\
\dots \arrow[r] & C_{n+1} \arrow[r, "\partial"] \arrow[d] & C_n \arrow[r, "\partial"] \arrow[d] & C_{n-1} \arrow[r] \arrow[d] & \dots \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Theorem 4.3.11 (Snake Lemma)**.**

Suppose

$$0_* \longrightarrow A_* \xrightarrow{f} B_* \xrightarrow{g} C_* \longrightarrow 0$$

is a short exact of chain complexes. Then, we have a long exact sequence

$$\begin{tikzcd}
 & \dots \arrow[r] & H_{n+1}(B_*) \arrow[r, "H_{n+1}(g)"] & H_{n+1}(C_*) \arrow[dlll, "\delta"'] \\
H_n(A_*) \arrow[r, "H_n(f)"'] & H_n(B_*) \arrow[r, "H_n(g)"] & H_n(C_*) \arrow[dll, "\delta"'] & , \\
H_{n-1}(A_*) \arrow[r, "H_{n-1}(f)"'] & H_{n-1}(B_*) \arrow[r] & \dots &
\end{tikzcd}$$

where the map δ is to be defined.

▶ *Proof.* This theorem is particularly tedious to prove. Hence, we only say how to define map δ and leave all the checking to the (interested) reader.

We will define $\delta : H_n(C_*) \to H_{n-1}(A_*)$. To do that, an element $[c] \in H_n(C_*)$ corresponds to a cycle $c \in C_n$ such that $\partial c = 0$. Take any $b \in B_n$ such that $g(b) = c$ (exists since g is surjective by exactness). Thus, $g(\partial b) = \partial c = 0$, so by exactness at B_{n-1}, we have $\partial b \in \operatorname{Im} f$, and so we may take (the unique) $a \in A_{n-1}$ such that $f(a) = \partial b$. Note that $f(\partial a) = \partial f(a) = \partial\partial b = 0$, so $\partial a = 0$ by exactness at A_{n-2}, which means that a is a cycle. Thus, we may take $\delta c = a$.

$$\begin{tikzcd}
 & B_n \arrow[r, "f"] \arrow[d, "\partial"] & C_n \arrow[d, "\partial"] \\
A_{n-1} \arrow[r, "f"] \arrow[d, "\partial"] & B_{n-1} \arrow[r, "g"] \arrow[d, "\partial"] & C_{n-1} \\
A_{n-2} \arrow[r, "f"] & B_{n-2} &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & b \arrow[r, mapsto, "g"] \arrow[d, mapsto, "\partial"] & c \arrow[d, mapsto, "\partial"] \\
a \arrow[r, mapsto, "f"] \arrow[d, mapsto, "\partial"] & \partial b \arrow[r, mapsto, "g"] \arrow[d, mapsto, "\partial"] & 0 \\
\partial a \arrow[r, mapsto, "f"] & 0 &
\end{tikzcd}$$

We have made an arbitrary choice of b, so we will show that the map δ does not depend on b. Suppose we have another choice $b' \in B_n$, so $g(b) = g(b')$, which means that $b - b' \in \operatorname{Ker} g$, so $b - b' \in \operatorname{Im} f$. Therefore, there exists $\alpha \in A_n$ such that $f(t) = b - b'$. Now, if a' is such that $f(a') = \partial b'$, then we have that

$$f(\partial t) = \partial(f(\alpha)) = \partial(b - b') = f(a) - f(a'), \quad \text{so} \quad \partial t = a - a',$$

(since f is injective). This means that $a - a'$ is in the boundary, so $[a] = [a'] \in H_{n-1}(A)$. ☐

Exercise 4.3.12. Complete the proof of snake lemma.

Remark 4.3.13 (Snake lemma is natural).

The construction of long exact sequence is natural. In particular, the diagram of chain complexes with exact rows

$$\begin{tikzcd} 0 \arrow[r] & A_* \arrow[r, "f"] \arrow[d, "\alpha"] & B_* \arrow[r, "g"] \arrow[d, "\beta"] & C_* \arrow[r] \arrow[d, "\gamma"] & 0 \\ 0 \arrow[r] & A'_* \arrow[r, "f'"] & B'_* \arrow[r, "g'"] & C'_* \arrow[r] & 0 \end{tikzcd}$$

induces a commuting diagram with exact rows

$$\begin{tikzcd} \dots \arrow[r] & H_n(A) \arrow[r, "H_n(f)"] \arrow[d, "H_n(\alpha)"] & H_n(B) \arrow[r, "H_n(g)"] \arrow[d, "H_n(\beta)"] & H_n(C) \arrow[r, "\delta"] \arrow[d, "H_n(\gamma)"] & H_{n-1}(A) \arrow[r] \arrow[d, "H_{n-1}(\alpha)"] & \dots \\ \dots \arrow[r] & H_n(A') \arrow[r, "H_n(f')"] & H_n(B') \arrow[r, "H_n(g')"] & H_n(C') \arrow[r, "\delta"] & H_{n-1}(A') \arrow[r] & \dots \end{tikzcd}$$

One common use of this diagram is to use Five Lemma (Theorem 4.3.4) to conclude that a downward arrow is an isomorphism.

§4.3.3 Relative Homology

We will now give the key application of Theorem 4.3.11. Working towards the goal of computing homology groups, we will, for any subspace $A \subseteq X$, we will relate the homology groups of A, of X, and of X/A (quotient by identifying everything in A to be the same). In order to do that, we will define the **relative homology group** $H_n(X, A)$. It will turn out that if $A \subseteq X$ is a "reasonable inclusion", then $H_n(X, A) = H_n(X/A)$.

Definition 4.3.14 (Inclusion of chain complexes).

An **inclusion of chain complexes** $C_* \subseteq D_*$ is a morphism

$$\begin{tikzcd} \dots \arrow[r, "\partial"] & C_n \arrow[r, "\partial"] \arrow[d, phantom, "\cap" rotate] & C_{n-1} \arrow[r, "\partial"] \arrow[d, phantom, "\cap"] & C_{n-2} \arrow[r, "\partial"] \arrow[d, phantom, "\cap"] & \dots \\ \dots \arrow[r, "\partial"] & D_n \arrow[r, "\partial"] & D_{n-1} \arrow[r, "\partial"] & D_{n-2} \arrow[r, "\partial"] & \dots \end{tikzcd}$$

Definition 4.3.15 (Quotient of chain complexes).

Given an inclusion of chain complexes, we can form the **quotient of chain complexes** D_*/C_* as

$$\dots \longrightarrow D_n/C_n \xrightarrow{\partial} D_{n-1}/C_{n-1} \xrightarrow{\partial} D_{n-2}/C_{n-2} \longrightarrow \dots$$

It is an assertion that this is well-defined.

Definition 4.3.16 (Relative homology).

Suppose that $A \subseteq X$ is a subspace, we define **relative homology**

$$H_n(X, A) = \text{homology of } S_*(X)/S_*(A).$$

For any space X, we have $H_n(X, \emptyset) = H_n(X)$.

Definition 4.3.17 (Category of Pairs)**.**

There is also a category $\mathbf{Top}_2$ with objects being pairs of topological spaces (A, X) such that $A \subseteq X$ with morphisms $(A, X) \to (B, Y)$ are continuous functions $f : X \to Y$ such that $f(A) \subseteq B$.

There are relative homology functors $H_n : \mathbf{Top}_2 \to \mathbf{Ab}$ and also functors $S_n : \mathbf{Top}_2 \to \mathbf{Ab}$ by $(X, A) \mapsto S_*(X)/S_*(A)$. The following proposition is a relative version of Theorem 4.2.8.

Proposition 4.3.18.

Suppose that $f, g : (X, A) \to (Y, B)$ are morphisms in $\mathbf{Top}_2$. If f and g are homotopic, then $H_n(f) = H_n(g)$.

Example 4.3.19 (The important long exact sequence)**.**

Suppose $A \subseteq X$ is the inclusion of a subspace, then we have the short exact sequence

$$0_* \longrightarrow S_*(A) \longrightarrow S_*(X) \longrightarrow S_*(X)/S_*(A) \longrightarrow 0_*,$$

so it turns to a long exact sequence

$$\begin{tikzcd}
 & \dots \arrow[r] & H_{n+1}(X) \arrow[r] & H_{n+1}(X, A) \arrow[dll, "\delta"] \\
 & H_n(A) \arrow[r] & H_n(X) \arrow[r] & H_n(X, A) \arrow[dll, "\delta"] \\
 & H_{n-1}(A) \arrow[r] & H_{n-1}(X) \arrow[r] & \dots
\end{tikzcd},$$

You will generally not need to access the map δ.

§4.4 Excision

The final result needed to compute homology groups is called **excision**, which we will spend several class going through the proof.

The result, in the most useful form, is as follows.

Theorem 4.4.1 (Excision I)**.**

If $A \subset X$ and U is an open subset of A such that $\overline{U} \simeq \text{interior}(A)$, then the inclusion map $X - U \to X$ induces isomorphism in homology groups

$$H_n(X - U, A - U) \simeq H_n(X, A)$$

for all $n \geq 0$.

This theorem implies the following more useful version.

Theorem 4.4.2 (Excision II)**.**

Consider a subspace $A \subset X$. Suppose there exists $B \subset X$ such that

(a) $\overline{A} \subseteq \text{interior}(B)$, and

(b) $A \subset B$ is a deformation retract.

(these are mild conditions and you are not expected to actually check it).

Then, for all integers $n \in \mathbb{Z}$, the quotient projection $X \to X/A$ (which sends A to a single point $\{b\}$) introduce an isomorphism

$$H_n(X, A) = H_n(X/A, \{b\})$$

for all $n \geq 0$.

We will prove both theorems by the end of this section.

§4.4.1 Relative Homology by a Point

In order to digest the result, we need to understand the relative homology w.r.t. a single point first. Suppose we have $b \in X$. What is the relation between $H_n(X)$ and $H_n(X, \{b\})$? First of all, we have a long exact sequence.

$$\begin{tikzcd}
 & & \cdots \arrow[dll, "\delta"'] \\
H_n(\{b\}) \arrow[r] & H_n(X) \arrow[r] & H_n(X, \{b\}) \arrow[dll, "\delta"'] \\
H_{n-1}(\{b\}) \arrow[r] & H_{n-1}(X) \arrow[r] & \cdots
\end{tikzcd}$$

- Suppose $n > 1$. Then, by inserting $H_n(\{b\}) = H_{n-1}(\{b\}) = 0$, we get an exact sequence

  $$0 \longrightarrow H_n(X) \longrightarrow H_n(X, \{b\}) \longrightarrow 0,$$

  so we deduce that

  $$H_n(X, \{b\}) \simeq H_n(X).$$

- Suppose $n = 0$. In this case, we have the right exact sequence

  $$\begin{tikzcd}
  H_0(\{b\}) \arrow[r] \arrow[d, phantom, "\wr|"] & H_0(X) \arrow[r] \arrow[d, phantom, "\wr|"] & H_0(X, \{b\}) \arrow[r] & 0 \\
  \mathbb{Z} & \mathbb{Z}\pi_0 X & &
  \end{tikzcd},$$

  and so we have

  $$H_0(X, \{b\}) \simeq \mathbb{Z}\pi_0 X/\mathbb{Z}.$$

- Suppose $n = 1$, then by plugging in $H_1(\{b\})$, $H_0(\{b\})$, and $H_0(X)$, we have the exact sequence

  $$0 \longrightarrow H_1(X) \xrightarrow{f} H_1(X, \{b\}) \xrightarrow{\delta} \mathbb{Z} \xrightarrow{g} \mathbb{Z}\pi_0 X$$

  The map g is induced from $\{b\} \hookrightarrow X$, and so it is injective and has zero kernel. Thus, $\operatorname{Im} \delta = \operatorname{Ker} g = 0$, and so δ is zero map. This means that f is bijective, and we have

  $$H_1(X, \{b\}) \simeq H_1(X).$$

In conclusion, we have the following result.

Proposition 4.4.3 (Relative homology to a point)**.**

Let b be a point in topological space X. Then,

$$H_n(X, \{b\}) = \begin{cases} H_n(X) & n \geq 1 \\ \mathbb{Z}\pi_0 X/\mathbb{Z} & n = 0. \end{cases}$$

In particular, combining excision (Theorem 4.4.2) and Proposition 4.4.3 gives

Corollary 4.4.4.

Consider a subspace $A \subset X$ that satisfies the conditions of Theorem 4.4.2. Then,

$$H_n(X, A) \simeq \begin{cases} H_n(X/A) & n \geq 1 \\ H_n(X/A)/\mathbb{Z} & n = 0. \end{cases}$$

§4.4.2 Example: Homology of S^1

Before delving into the proof of excision, let us demonstrate an example application.

Let $X = [0, 1]$ and $A = \{0, 1\}$. Notice that $X/A \simeq S^1$. We have the long exact sequence:

$$\begin{tikzcd} \dots \arrow[r] & H_n(\{0,1\}) \arrow[r] & H_n([0,1]) \arrow[r] & H_n(X, A) \arrow[dll] \\ & H_{n-1}(\{0,1\}) \arrow[r] & H_{n-1}([0,1]) \arrow[r] & \dots \end{tikzcd}$$

- Obviously, we have $H_0(S^1) = \mathbb{Z}$, and so $H_0(X, A) = 0$.
- When $n \geq 2$, we fill the information out as follows:

$$\begin{tikzcd} \dots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & H_n(S^1) \arrow[dll] \\ & 0 \arrow[r] & 0 \arrow[r] & \dots \end{tikzcd},$$

  and this implies $H_n(S^1) = 0$ for all $n \geq 2$.
- When $n = 1$, we fill the diagram above out as follows.

$$\begin{tikzcd} \dots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & H_1(S^1) \arrow[dll] \\ & \mathbb{Z} \oplus \mathbb{Z} \arrow[r] & \mathbb{Z} \arrow[r] & \dots \end{tikzcd}$$

  the last map sends $(1, 0) \mapsto 1$ and $(0, 1) \mapsto 1$, and so it has kernel $\mathbb{Z}$. Since the map $H_1(S^1) \to \mathbb{Z} \oplus \mathbb{Z}$ is injective, it follows that $H_1(S^1) \simeq \mathbb{Z}$.

Thus, we have

Proposition 4.4.5.

We have

$$H_k(S^1) = \begin{cases} \mathbb{Z} & k = 0, 1 \\ 0 & k \geq 2. \end{cases}$$

§4.4.3 Excision I implies Excision II

In this section, we prove that Excision I (Theorem 4.4.1) implies Excision II (Theorem 4.4.2). First, we prove the following propositon.

> **Proposition 4.4.6.**
>
> Suppose $A \subset B \subset X$ and that $A \subset B$ is a deformation retract. Then, $H_n(X, A) \simeq H_n(X, B)$.

▶ *Proof.* There is a morphism $(X, A) \to (X, B)$ in $\mathbf{Top}_2$ by inclusion. Then, applying the snake lemma, we find the diagram

$$\begin{tikzcd} H_n(A) \arrow[r] \arrow[d, "\simeq"] & H_n(X) \arrow[r] \arrow[d, "\mathrm{id}"] & H_n(X,A) \arrow[r, "\delta"] \arrow[d] & H_{n-1}(A) \arrow[r] \arrow[d, "\simeq"] & H_{n-1}(X) \arrow[d, "\mathrm{id}"] \\ H_n(B) \arrow[r] & H_n(X) \arrow[r] & H_n(X,B) \arrow[r, "\delta"] & H_{n-1}(B) \arrow[r] & H_{n-1}(X) \end{tikzcd},$$

where all squares commute because the construction in Snake lemma is functorial. We now use Five Lemma. In order to prove that the middle map is an isomorphism, we note that

- The first and fourth map are isomorphisms because $A \subset B$ is deformation retract.
- The second and fifth map is simply the identity map.

Thus, all downward-facing arrows except the middle are isomorphisms. Hence, the middle arrow $H_n(X, A) \to H_n(X, B)$ is an isomorphism by Five Lemma. Hence, we are done. □

▶ *Proof of Theorem 4.4.2 assuming Theorem 4.4.1.* Consider the commuting diagram of map between pairs of topological spaces.

$$\begin{tikzcd} (X,A) \arrow[r, "(2)"] \arrow[d] & (X,B) \arrow[d] & (X-A, B-A) \arrow[l, "(3)"'] \arrow[d, "(1)"] \\ (X/A, \{b\}) \arrow[r, "(2)"] & (X/A, B/A) & (X/A-\{b\}, B/A-\{b\}) \arrow[l, "(3)"'] \end{tikzcd}$$

When applying homology groups, note that

- the arrow marked (1) becomes an isomorphism because it is homotopy equivalent to identity map (using Proposition 4.3.18).
- the arrows marked (2) become isomorphisms because $A \subset B$ and $\{b\} \subset B/A$ are both deformation retract, hence we may apply the previous Proposition 4.4.6.
- the arrows marked (3) become isomorphisms because we can use Excision 2.0 (Theorem 4.4.1).
- Finally, by composing, we see that all the remaining arrows are isomorphisms. □

§4.4.4 Locality Principle

The next two subsections will prove Excision I (Theorem 4.4.1). In this section, we will establish what is called the locality principle, which is has a nice and intuitive statement that we will only need it en route to proving excision.

We first begin with the following basic definitions.

Definition 4.4.7 (Cover and small simplices)**.**

A **cover** of a space X is a set $\mathcal{A}$ of subsets of X such that

$$X = \bigcup_{A \in \mathcal{A}} \operatorname{interior}(A).$$

A simplex $\sigma : \Delta^n \to X$ is said to be **$\mathcal{A}$-small** if it factors as a composite map $\Delta^n \to A \hookrightarrow X$ for some $A \in \mathcal{A}$. Define $S_n^{\mathcal{A}}(X) \subseteq S_n(X)$ as the free abelian group generated by $\mathcal{A}$-small simplices.

If σ is $\mathcal{A}$-small, then $\partial\sigma \in S_{n-1}^{\mathcal{A}}(X)$. Hence, the chain complex $S_*(X)$ induces a sub-chain complex $S_*^{\mathcal{A}}(X)$.

$$\begin{array}{ccccccccc}
\dots & \xrightarrow{\partial} & S_n(X) & \xrightarrow{\partial} & S_{n-1}(X) & \xrightarrow{\partial} & S_{n-2}(X) & \xrightarrow{\partial} & \dots \\
 & & \cup\!| & & \cup\!| & & \cup\!| & & \\
\dots & \xrightarrow{\partial} & S_n^{\mathcal{A}}(X) & \xrightarrow{\partial} & S_{n-1}^{\mathcal{A}}(X) & \xrightarrow{\partial} & S_{n-2}^{\mathcal{A}}(X) & \xrightarrow{\partial} & \dots
\end{array}$$

We now state the **locality principle**.

Theorem 4.4.8 (Locality Principle)**.**

If X is a topological space with cover $\mathcal{A}$ then the inclusion $S_n^{\mathcal{A}}(X) \to S_n(X)$ induces isomorphism on homology groups.

The intuitive idea of this theorem is that **any simplex $\sigma : \Delta^n \to X$ can be written, up to boundaries, as a sum of smaller simplices**.

To make this precise, we will define a natural transformation

$$\$: S_n(X) \to S_n(X)$$

known as the **barycentric subdivision** that we will define by induction on n. It is defined by what it does on a map $\sigma^n \to X$. For it to be natural transformation, we need the following diagram to commute:

$$\begin{tikzcd}
S_n(\Delta^n) \arrow[r, "\$"] \arrow[d, "S_n(\sigma)"] & S_n(\Delta^n) \arrow[d, "S_n(\sigma)"] \\
S_n(X) \arrow[r, "\$"] & S_n(X).
\end{tikzcd}$$

In particular, tracing through the diagram starting with id_{Δ^n} yields

$$\begin{tikzcd}
\mathrm{id}_{\Delta^n} \arrow[r, mapsto] \arrow[d, mapsto] & \$(\mathrm{id}_{\Delta^n}) \arrow[d, mapsto] \\
\sigma \arrow[r, mapsto] & \$\sigma
\end{tikzcd},$$

so **$\$\sigma$ is in fact uniquely determined by $\$(\mathrm{id}_{\Delta^n})$**. We thus only need to define what $\$$ does on a simplex, which we do as follows:

- For $n = 0$, we set
$$\$(\mathrm{id}_{\Delta^0}) = \mathrm{id}_{\Delta^0}.$$
- For general n, consider a simplex Δ^n generated by basis $e_0, e_1, \dots, e_n$. We let
$$b = \frac{e_0 + e_1 + \dots + e_n}{n+1}$$

be the center of Δ^n. As seen in the proof of Theorem 4.2.7, we now define the map

$$\begin{aligned} b^* &: S_{n-1}(\Delta^n) \to S_n(\Delta^n) \\ (b^*\sigma_{n-1})(t_0, t_1, \ldots, t_n) &= t_0 b + (1-t_0)\,\sigma_{n-1}\left(\frac{t_1}{1-t_0}, \ldots, \frac{t_n}{1-t_0}\right). \end{aligned}$$

and finally, we set

$$\$(\mathrm{id}_{\Delta^n}) = b^*\$(\partial \mathrm{id}_{\Delta^n}).$$

Let us draw an example for $n = 1$ and $n = 2$.

- For $n = 1$, we have

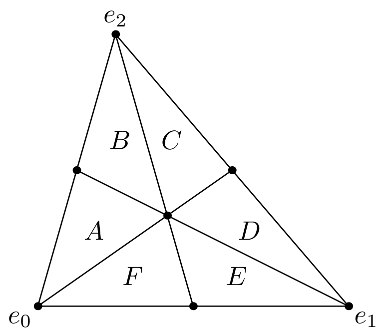
  $$\$(\mathrm{id}_{\Delta^1}) = f - g,$$

  where

  $$\begin{aligned} f &: [0,1] \to [0,1] \text{ defined by } f(t) = t/2 \\ g &: [0,1] \to [0,1] \text{ defined by } g(t) = 1 - t/2, \end{aligned}$$

  so f is a path from 0 to $1/2$, while g is a path from 1 to $1/2$. In particular, we are just subdividing a segment from $[0,1]$ into two segments.

- For $n = 2$, essentially what we are doing is that we first subdivide the boundary. Then, we take the center and then connect to the subdivision of boundary. We get the following picture:

  and we have $\$(\mathrm{id}_{\Delta^2}) = A - B + C - D + E - F$.

Proposition 4.4.9 (\$ is chain homotopic to identity map)**.**

The map $\$: S_n(X) \to S_n(X)$ for various n assembles to a chain map $\$: S_*(X) \to S_*(X)$, which is naturally chain homotopic to $\mathrm{id}_{S_*(X)}$.

▶ *Proof.* In the proof of Theorem 4.2.7, we have shown that b^* is a chain homotopy from $\eta\varepsilon$ to id. In particular, for any $\sigma : \Delta^n \to X$,

$$\partial b^*(\sigma) + b^*\partial(\sigma) = \sigma - \eta(\varepsilon(\sigma)) = \begin{cases} \sigma - b & \text{if } n = 0 \\ \sigma & \text{if } n \geq 1. \end{cases}$$

We first check that \$ is a chain map by showing that it commutes with ∂. that $\partial(\$(\sigma)) = \$(\partial(\sigma))$ for every $\sigma : \Delta^n \to X$. We do this by induction on n. The case that $n = 0$ is clear. For $n \geq 1$, we compute

$$\begin{aligned} \partial(\$(\sigma)) &= \partial(b^*\$(\partial\sigma)) \\ &= \$\partial(\sigma) - b^*\partial\$(\partial\,\sigma) \\ &= \$\partial(\sigma) - b^*\$(\partial\partial\sigma) && \text{(inductive hypothesis)} \\ &= \$\partial(\sigma) && (\partial^2 = 0) \end{aligned}$$

as desired.

Now, we establish the chain homotopy to the identity map. We use the same trick as Theorem 4.2.8 Once again, we use induction on n to define a natural chain homotopy $h_X : S_n(X) \to S_{n+1}(X)$. For the base case $n = 0$,

For the inductive step, we want our h to satisfy

$$\partial h_X(\sigma) = -h_X \partial(\sigma) + \$(\sigma) - \sigma.$$

We define h_X to be natural in X, so it is uniquely determined by $h_{\Delta^n}(\mathrm{id}_{\Delta^n})$ for each n. Again, we define it inductively on n, with the base case $n = 0$ being trivial. It suffices to show that

$$\partial h_{\Delta^n}(\mathrm{id}_{\Delta^n}) = -h_{\Delta^n}\partial(\mathrm{id}_{\Delta^n}) + \$(\mathrm{id}_{\Delta^n}) - \mathrm{id}_{\Delta^n},$$

and naturality will do the rest. To do this, we take $h_{\Delta^n}(\mathrm{id}_{\Delta^n})$ to be the preimage of the right hand side. To show that this exists, it suffices to show that the right side is a boundary. However, since $H_{n-1}(\Delta^n) = 0$, any boundary is a cycle, so it suffices to show that ∂ of the right hand side is 0. To do this, we compute

$$\begin{aligned}
&\partial\Big(-h_{\Delta^n}\partial(\mathrm{id}_{\Delta^n}) + \$(\mathrm{id}_{\Delta^n}) - \mathrm{id}_{\Delta^n}\Big)\\
&= -\partial(h_{\Delta^n}\partial(\mathrm{id}_{\Delta^n})) + \partial(\$(\mathrm{id}_{\Delta^n})) - \partial(\mathrm{id}_{\Delta^n})\\
&= \Big(h_{\Delta^n}(\partial(\partial(\mathrm{id}_{\Delta^n}))) - \$(\partial(\mathrm{id}_{\Delta^n})) + \partial(\mathrm{id}_{\Delta^n})\Big) + \partial(\$(\mathrm{id}_{\Delta^n})) - \partial(\mathrm{id}_{\Delta^n})\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(induction hypothesis)}\\
&= h_{\Delta^n}(\partial(\partial(\mathrm{id}_{\Delta^n})))\\
&= 0, \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(since } \partial^2 = 0\text{)}
\end{aligned}$$

which imply that we can select the desired $h_{\Delta^n}(\mathrm{id}_{\Delta^n})$. ☐

Proposition 4.4.10.

If $\sigma : \Delta^n \to X$ is an element on $\mathrm{Sing}_n(X)$ and $\mathcal{A}$ is a cover of X, then

$$\$^k(\sigma) \in \mathrm{Sing}_n^{\mathcal{A}}(X)$$

for some integer $k \geq 0$.

▶ *Proof.* First, consider the case where $X = \Delta^n$ and $\sigma = 1_{\Delta^n}$. By Lebesgue Covering Lemma (Problem 1.F), take $\varepsilon > 0$ such that for all $x \in \Delta^n$, $B_\varepsilon(x)$ is contained in U for some $U \in \mathcal{A}$. By some geometric calculation, \$ reduces the diameter to a factor of at most $\frac{n}{n+1}$, so iterated applications of subdivision $\$^k$ makes the diameter of every simplex at most ε. Thus, every simplex is contained in U for some $U \in \mathcal{A}$.

In general, given $\sigma : \Delta^n \to X$ and $\mathcal{U} = \{\sigma^{-1}(A) : A \in \mathcal{A}\}$ will be a cover of Δ^n. Thus, we can choose k such that all simplices of $\$(\mathrm{id}_{\Delta^n})$ are contained in $\mathcal{U}$. ☐

Once we have tool to subdivide simplices, we will now prove the locality principle (Theorem 4.4.8).

▶ *Proof of Theorem 4.4.8.* We verify directly that the map is bijective.

▶ **Surjectivity** Suppose that $\alpha \in Z_n(X)$, so $\partial A = 0$. We let T be the chain homotopy between $\$^k$ and $\mathrm{id}_{S_*(X)}$. Then, by Proposition 4.4.9, we have

$$\$^k\alpha - \alpha = \partial T\alpha - T\partial\alpha = \partial T\alpha.$$

Since $\$^k\alpha \in Z_n^{\mathcal{A}}(X)$ and $\partial T\alpha \in B_n^{\mathcal{A}}(X)$, surjectivity follows.

▶ **Injectivity** Suppose $\alpha \in Z_n^{\mathcal{A}}(X)$ has a trivial image in $H_n(X)$, so $\alpha = \partial\beta$. We choose k such that $\$^k\beta$ is $\mathcal{A}$-small. Then, we have

$$\begin{aligned}\partial(\$^k\beta) - \alpha &= \partial(\$^k - 1)\beta \\ &= \partial(\partial T + T\partial)\beta \\ &= \partial T\partial\beta && \text{(Proposition 4.4.9)} \\ &= \partial T\alpha,\end{aligned}$$

We have $\partial(\$^k\beta) \in B_n^{\mathcal{A}}(X)$ and we note that if α is $\mathcal{A}$-small, we also have (by its construction which is left out) that $T\alpha$ is $\mathcal{A}$-small. Hence, $\partial T\alpha \in B_n^{\mathcal{A}}(X)$. Thus, $\alpha \in B_n^{\mathcal{A}}(X)$, so α also has a trivial image. □

§4.4.5 Proof of Excision I

We now prove Excision I, Theorem 4.4.1. For convenience, we state the theorem below.

Theorem 4.4.11 (Theorem 4.4.1, restatement)**.**

If $U \subseteq A \subseteq X$ is such that $\overline{U} \subseteq$ interior A, then

$$H_n(X, A) \simeq H_n(X - U, A - U)$$

for all $n \in \mathbb{Z}$.

▶ *Proof.* Let $\mathcal{A} = \{A, X - U\}$. Consider the following commuting diagram.

$$\begin{tikzcd} 0 \arrow[r] & S_*(A) \arrow[r] \arrow[d] & S_*^{\mathcal{A}}(X) \arrow[r] \arrow[d, hook] & S_*^{\mathcal{A}}(X)/S_*(A) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & S_*(A) \arrow[r] & S_*(X) \arrow[r] & S_*(X)/S_*(A) \arrow[r] & 0 \end{tikzcd},$$

so we get a long exact sequence of homology groups

$$\begin{tikzcd} H_n(A) \arrow[r] \arrow[d, "\mathrm{id}"] & H_n^{\mathcal{A}}(X) \arrow[r] \arrow[d, "\text{locality}"] & H_n\left(S_*^{\mathcal{A}}(X)/S_*(A)\right) \arrow[r, "\delta"] \arrow[d] & H_{n-1}(A) \arrow[r] \arrow[d, "\mathrm{id}"] & H_{n-1}^{\mathcal{A}}(X) \arrow[d, "\text{locality}"] \\ H_n(A) \arrow[r] & H_n(X) \arrow[r] & H_n(X, A) \arrow[r, "\delta"] & H_{n-1}(A) \arrow[r] & H_{n-1}(X) \end{tikzcd}$$

The first and fourth maps are identity maps. The second and fifth maps are isomorphisms by locality principle. Thus, by Five Lemma, the third map is an isomorphism, which means that

$$H_n(S_*^{\mathcal{A}}(X)/S_*(A)) \simeq H_n(X, A). \tag{4.3}$$

However, we note that

$$\frac{S_n^{\mathcal{A}}(X)}{S_*(A)} = \frac{S_n(A) + S_n(X - U)}{S_n(A)} \simeq \frac{S_n(X - U)}{S_n(A) \cap S_n(X - U)} \simeq \frac{S_n(X - U)}{S_n(A - U)},$$

where the second equality uses the second isomorphism theorem ($(M + N)/N \simeq M/(M \cap N)$). Hence, we deduce that

$$H_n(S_*^{\mathcal{A}}(X)/S_*(A)) \simeq H_n(X - U, A - U) \tag{4.4}$$

Combining (4.3) and (4.4) yields the result. □

§4.5 Problems

See the next chapter for explicit calculations of homology. You can attempt some of the problems there, but it will get easier once you learn more computational tools.

Problem 4.A (Splitting lemma)**.** Suppose that we have an exact sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0.$$

Then, prove that the following are equivalent:

(i) $B \simeq A \oplus C$.

(ii) There exists a map $s : C \to B$ such that $g \circ s = \mathrm{id}_C$.

(iii) There exists a map $t : B \to A$ such that $t \circ f = \mathrm{id}_A$.

In particular, these always holds when $C = \mathbb{Z}^r$! This is a useful tool to determine the exact sequence.

Problem 4.B (Suspension Theorem)**.** For any topological space X, define its **suspension** ΣX to be the quotient of $X \times [-1, 1]$, where we identify $X \times \{-1\}$ to one point and $X \times \{1\}$ to another. Prove that for any $n \geq 1$,

$$H_n(\Sigma X) = \begin{cases} H_{n-1}(X) & \text{if } n \geq 2 \\ H_0(X)/\mathbb{Z} & \text{if } n = 1 \end{cases}.$$

Problem 4.C (Reversing simplex)**.** Let X be a topological space. Define a map $R : S_n(X) \to S_n(X)$ by

$$R(\sigma) = (-1)^{n(n-1)/2}[\sigma(v_n), \ldots, \sigma(v_0)].$$

for any $\sigma : \Delta^n \to X$, where $v_0, \ldots, v_n$ are vertices of standard n-simplex. In other words, $R(\sigma)$ is $(-1)^{n(n-1)/2}$ times the simplex obtained by reversing the order of vertices of the simplex corresponding to σ.

Prove that R is chain-homotopic to the identity map.

Problem 4.D (Hurewicz's Theorem)**.** For any group G, the **abelianization** G^{ab} is the quotient group $G/[G, G]$, where

$$[G, G] = \text{subgroup generated by } \{aba^{-1}b^{-1} : a, b \in G\}.$$

In some sense, G^{ab} is the smallest abelian quotient of G.

Prove that for any path-connected topological space X, we have $H_1(X) \simeq \pi_1(X)^{\mathrm{ab}}$.

5 Computations and Applications of Homology

Having most of the technical setup of homology groups done, we will now explore more tools that will allow us to actually compute the homology groups of any topological space that we would like.

We begin by presenting Mayer-Vietoris sequence, which allows us to determine homology of X by cutting it to two open subspaces A and B such that $X = A \cup B$, and then determine $H_n(X)$ from $H_n(A)$, $H_n(B)$, and $H_n(A \cap B)$. We use this to compute homology of a sphere, and record some consequences.

Then, we revisit the introduction in the previous chapter, in which we *compute* homology by cutting the topological space into edges, triangles, etc. To do this, we define, from a way to cut the topological space X, a simpler chain complex that has the same homology groups as the singular chain complex of X. This complex is finitely-generated, and thus allows us to compute homology of any CW complex almost combinatorially.

§5.1 Mayer-Vietoris Sequence

We now introduce another powerful computational tool that allows us to compute homology groups more easily.

Theorem 5.1.1 (Mayer-Vietoris Sequence).

Suppose that X is a topological space and $A, B \subset X$ are open sets such that $X = A \cup B$. Then, there is a long exact sequence

$$\begin{tikzcd}
\dots \arrow[r] & H_n(A \cap B) \arrow[r, "H_n(i) \oplus H_n(j)"] & H_n(A) \oplus H_n(B) \arrow[r, "H_n(k) - H_n(\ell)"] & H_n(X) \arrow[dll, "\delta"] & \\
& H_{n-1}(A \cap B) \arrow[r] & H_{n-1}(A) \oplus H_{n-1}(B) \arrow[r] & H_{n-1}(X) \arrow[r] & \dots
\end{tikzcd}$$

where i, j, k, l denote inclusions $i : A \cap B \to A$, $j : A \cap B \to B$, $k : A \to X$, and $\ell : B \to X$.

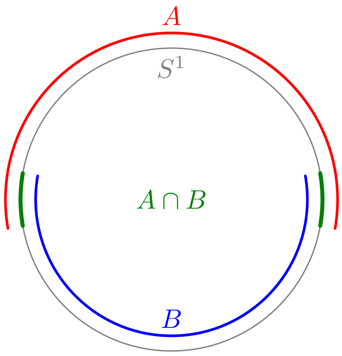
▶ *Proof.* Let $\mathcal{A} = \{A, B\}$. There is the exact sequence of chain complex

$$\begin{array}{ccccccccc}
0 \longrightarrow & S_*(A \cap B) & \longrightarrow & S_*(A) \oplus S_*(B) & \longrightarrow & S_*^{\mathcal{A}}(X) & \longrightarrow 0 \\
& x & \mapsto & (x, x) & & & , \\
& & & (x, y) & \mapsto & x - y &
\end{array}$$

which by Theorem 4.3.11 induces the desired long exact sequence. The third factor has homology groups isomorphic to those of X because of locality principle (Theorem 4.4.8). □

Let's demonstrate this by computing homology group of S^1.

▶ **Homology Groups of S^1.** We now revisit homology group of a circle, this time using Mayer Vietoris Sequence. Let $X = S^1$ and A and B are two overlapping open intervals.



By Mayer-Vietoris (Theorem 5.1.1), we have the exact sequence

$$H_n(A \cap B) \longrightarrow H_n(A) \oplus H_n(B) \longrightarrow H_n(X) \longrightarrow H_{n-1}(A \cap B) \longrightarrow H_{n-1}(A) \oplus H_{n-1}(B).$$

We split into some cases.

- **If $n \geq 2$**, then we fill the diagram above to get

  $$0 \longrightarrow 0 \longrightarrow H_n(X) \longrightarrow 0 \longrightarrow 0,$$

  and so $H_n(S^1) = 0$ for all $n \geq 2$.

- **If $n = 0$**, then we get the exact sequence

  $$\mathbb{Z} \oplus \mathbb{Z} \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \longrightarrow H_0(S^1) \longrightarrow 0 \longrightarrow 0.$$

  The first map is $(1, 0) \mapsto (1, 1)$ and $(0, 1) \mapsto (1, 1)$, so the image is $\mathbb{Z}$. Thus, $H_0(S^1) \simeq \mathbb{Z}$. Note that this is consistent with Exercise 4.2.2.

- **If $n = 1$**, then we fill the diagram above to get

  $$0 \longrightarrow 0 \longrightarrow H_1(S^1) \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \longrightarrow \mathbb{Z} \oplus \mathbb{Z}.$$

  The kernel of last map is $\{(a, -a) : a \in \mathbb{Z}\}$. Thus, $H_1(S^1) \simeq \mathbb{Z}$.

Thus, we get the same result as done by relative homology.

§5.2 Homology of Spheres

We will now do the first nontrivial example of homology: homology of a sphere. In particular, we will show the following:

Proposition 5.2.1 (Homology of a sphere).

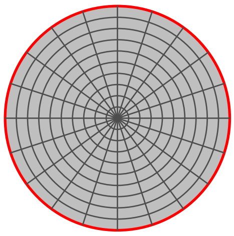
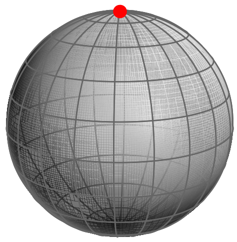
For any $n \geq 1$, we have

$$H_k(S^n) = \begin{cases} \mathbb{Z} & k = 0 \text{ or } n \\ 0 & \text{otherwise.} \end{cases}$$

Obviously, $H_0(S^n) = \mathbb{Z}$ because S^n is path-connected. Thus, we will focus on the computation of $H_k(S^n)$ for $k > 0$. In fact, we will do this in two ways: by relative homology, and by Mayer Vietoris.

§5.2.1 Homology of Spheres in Two Ways

▶ By relative homology Recall that a sphere S^n can be viewed as a disk D^n, identifying the boundary as a single point. Here is a picture for $n = 2$.

In particular, if $X = D^n$ and $A = S^{n-1}$ on the boundary, then $X/A \simeq S^n$, and so by excision (Theorem 4.4.2),

$$H_k(S^n) = H_k(X, A) \quad \text{for all } k > 0.$$

We also recall that since X is a convex set, $H_k(X) = 0$ for all $k > 0$. Moreover, we will use induction on n, hence we assume the induction hypothesis

$$H_k(A) = \begin{cases} \mathbb{Z} & k = 0, n-1 \\ 0 & \text{otherwise} \end{cases}.$$

Now, by Theorem 4.3.11, we have the long exact sequence

$$H_k(A) \longrightarrow H_k(X) \longrightarrow H_k(X, A) \longrightarrow H_{k-1}(A) \longrightarrow H_{k-1}(X).$$

We have a couple cases.

- **If $k = 1$**, then filling this diagram gives
$$0 \longrightarrow 0 \longrightarrow H_1(S^n) \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z},$$
and since the last map is injective, it follows that $H_1(S^n) = 0$.
- **If $k = n$**, then filling this diagram gives
$$0 \longrightarrow 0 \longrightarrow H_n(S^n) \longrightarrow \mathbb{Z} \longrightarrow 0,$$
where we use the induction hypothesis $H_{n-1}(S^{n-1}) = \mathbb{Z}$. Thus, we conclude that $H_n(S^n) \simeq H_{n-1}(S^{n-1}) \simeq \mathbb{Z}$.
- **If $k \neq 1, n$**, then every factor is 0 except $H_k(X, A) \simeq H_k(S^n)$, so $H_k(S^n) = 0$.

This proves Proposition 5.2.1.

▶ By Mayer-Vietoris A sphere S^n can be viewed as the top hemisphere A and the bottom hemisphere B such that

$$A \simeq D^n, \quad B \simeq D^n, \quad \text{and} \quad A \cap B \simeq S^{n-1} \times [0, 1].$$

which is homotopy equivalent to S^{n-1}.

Therefore, by Mayer-Vietoris (Theorem 5.1.1), we have the long exact sequence

$$H_k(A \cap B) \longrightarrow H_k(A) \oplus H_k(B) \longrightarrow H_k(X) \longrightarrow H_{k-1}(A \cap B) \longrightarrow H_{k-1}(A) \oplus H_{k-1}(B).$$

We have a couple cases.

- **If $k = 1$,** then we fill the diagram to get

$$0 \longrightarrow 0 \longrightarrow H_1(S^n) \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \oplus \mathbb{Z}.$$

The last map is injective, because it sends $1 \in H_0(A \cap B)$ to $(1, 1) \in H_0(A) \oplus H_0(B)$. Therefore, $H_1(S^n) = 0$.

- **If $k = n$,** then we fill the diagram to get

$$0 \longrightarrow 0 \longrightarrow H_n(S^n) \longrightarrow H_{n-1}(S^{n-1}) \longrightarrow 0,$$

so we have that $H_n(S^n) \simeq H_{n-1}(S^{n-1}) \simeq \mathbb{Z}$, by induction hypothesis.

- **If $k \neq 1, n$,** then every factor is 0 except $H_k(S^n)$, so $H_k(S^n) = 0$.

This proves Proposition 5.2.1.

§5.2.2 Consequences

We now record some consequences of Proposition 5.2.1.

Corollary 5.2.2 (Invariance of dimension).

If $p \neq q$, then $\mathbb{R}^p$ is not homeomorphic to $\mathbb{R}^q$.

▶ *Proof.* If they were, then $\mathbb{R}^p \setminus \{\text{point}\}$ and $\mathbb{R}^q \setminus \{\text{point}\}$ are also homeomorphic. However, $\mathbb{R}^p \setminus \{\text{point}\} \simeq S^{p-1}$ and $\mathbb{R}^q \setminus \{\text{point}\} \simeq S^{q-1}$, which have different homology groups by Proposition 5.2.1, so they are not even homotopy equivalent. □

Theorem 5.2.3 (Brouwer Fixed Point Theorem).

Let D^n be the n-dimensional disk. Suppose $f : D^n \to D^n$ is continuous, then f has a fixed point.

▶ *Proof.* Suppose not. Consider the function $g : D^n \to S^{n-1}$ by $g(x)$ being the intersection of ray joining $f(x)$ to x with the boundary of the circle. Thus, we have the composition

$$S^{n-1} \hookrightarrow D^n \xrightarrow{g} S^{n-1},$$

and applying the H_{n-1} functor gives a map

$$\begin{array}{ccccc} H_{n-1}(S^{n-1}) & \longrightarrow & H_{n-1}(D^n) & \longrightarrow & H_{n-1}(S^{n-1}) \\ \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & \mathbb{Z}. \end{array}$$

Since f sends the boundary to the boundary, we get that $g(x) = x$ for every x on the boundary. Therefore, the overall composition above must be the identity map. However, it factors through 0, so it must be 0. This is a contradiction. □

§5.2.3 Degree of a Map on Spheres

Definition 5.2.4 (Degree).

The **degree** $\deg f$ of continuous map $f : S^n \to S^n$ is the image of 1 under

$$H_n(f) : H_n(S^n) \to H_n(S^n);$$

notice that both groups are $\mathbb{Z}$ by Proposition 5.2.1. In other words, $H_n(f)$ is a multiplication by $\deg f$ map.

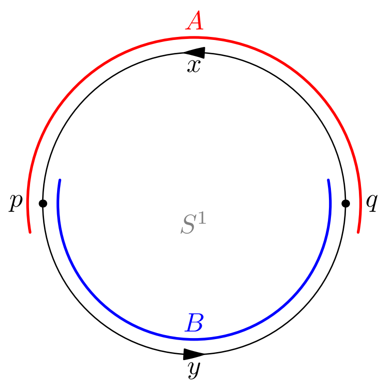
Here are some simple properties that immediately follows from the definition.

- $\deg(\mathrm{id}_{S^n}) = 1$.
- $\deg(g \circ f) = \deg(g)\deg(f)$.
- f is a homeomorphism if and only if $\deg f = \pm 1$.

Example 5.2.5.

One expects the degree of the reflection map $f : S^1 \to S^1$ to be -1. However, let us argue it correctly.

As in the computation of $H_1(S^1)$ via Mayer-Vietoris sequence in Section 5.1, we let A be the top half of the S^1 and B be the bottom half. Let p and q be points on different components of $A \cap B$. Let x and y be the path as in the following diagram.



Then we have an injective map $H_1(S^1) \to H_0(A) \oplus H_0(B)$, this is the boundary map, so we have to go through its construction described in the proof of Theorem 4.3.11.

First, recall the diagram

$$\begin{array}{ccccc}
S_1(A \cap B) & \longrightarrow & S_1(A) \oplus S_1(B) & \longrightarrow & S_1(S^1) \\
\downarrow & & \downarrow & & \downarrow \\
S_0(A \cap B) & \longrightarrow & S_0(A) \oplus S_0(B) & \longrightarrow & S_0(S^1)
\end{array}$$

Take a cycle $x + y \in S_1(S^1)$. Then one choice of preimage is $(x, -y) \in S_1(A) \oplus S_1(B)$. Taking boundary map, this is sent to $(p - q, p - q) \in S_0(A) \oplus S_0(B)$. A preimage of this is $p - q \in H_0(A \cap B)$. In particular, a counterclockwise loop is sent to $p - q$.

Similarly, a clockwise loop is sent to $q - p$. Since reflection sends a counterclockwise to a clockwise loop, the reflection indeed has degree -1.

Note that this calculation will be much easier when we learn about cellular homology in the next section (Section 5.3).

In general, a reflection in some plane through the origin in $\mathbb{R}^{n+1}$ restricts to a self-map on S^n with degree -1. In short, reflection has degree -1. This can be shwon by a Mayer-Vietoris calculation similar to above.

Similar to reflection map is the **antipodal map** from $S^n \to S^n$, which sends x to $-x$. One can compute its degree directly with complexes. However, a simpler way is to note that the antipodal map in $S^n \to S^n$ is a composition of $n+1$ reflections. For example, if $n=2$, then we have the composition

$$(x,y,z) \mapsto (-x,y,z) \mapsto (-x,-y,-z) \mapsto (-x,-y,-z),$$

so the antipodal map $S^n \to S^n$ has degree $(-1)^{n+1}$.

The following proposition summarizes what we discuss so far.

Proposition 5.2.6 (Properties of Degrees)**.**

We have the following.

(a) The reflection map on S^n has degree -1.

(b) The antipodal map on S^n has degree $(-1)^{n+1}$.

By homotopy invariance (Theorem 4.2.8), homotopic maps have the same degree. This can be useful in calculating degrees. For example, consider the following lemma.

Lemma 5.2.7.

If a map $f : S^n \to S^n$ has no fixed point, then $\deg f = (-1)^{n+1}$.

▶ *Proof.* We construct a homotopy between f and the antipodal map. If we represent S^n by $\{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$, then the homotopy is

$$h_t(x) = \frac{(1-t)f(x) - tx}{\|(1-t)f(x) - tx\|}.$$

The denominator is 0 if and only if $(1-t)f(x) = tx$. Comparing magnitude reveals that this is equivalent to $t = \frac{1}{2}$ and so $f(x) = x$, which is impossible. Therefore, the map h_t is continuous. We have $h_0(x) = f(x)$ and $h_1(x) = -x$, so h indeed forms a homotopy between f and the antipodal map. □

We now present a classic theorem. In the case of $n=2$, it says that "there is no way to comb a hair in a ball".

Theorem 5.2.8 (Hairy Ball Theorem)**.**

Let n be an even positive integer. Then, there is no continuous non-vanishing tangent vector field on S^n. In other words, if the sphere is centered at 0, there is no continuous map $f : S^n \to S^n$ such that $x \cdot f(x) = 0$ (where $\cdot$ denote the standard dot product).

▶ *Proof.* The map f has no fixed point (why?), so by Lemma 5.2.7, it has degree $(-1)^{n+1}$.

However, we will construct another homotopy between f and the identity map.

$$h_\theta(x) = \cos\theta\, f(x) + \sin\theta\, x,$$

which always lies on the unit sphere. We have $h_0(x) = f(x)$ and $h_{\pi/2}(x) = x$, so this provides a homotopy between f and the identity map. Hence, f has degree 1, contradicting the previous paragraph. □

Remark 5.2.9.

Two homotopic maps $S^n \to S^n$ have the same degree. The converse of this also holds: any two maps with the same degree are homotopic. This is equivalent to the fact that the homotopy group $\pi_n S^n$ is $\mathbb{Z}$, which is much harder to prove.

§5.3 Cellular Homology

We have now established basic properties of homology that allows us to compute various homology groups. The next goal is to revisit the introduction where instead of using *all* simplices to compute homology, we *cut* them. The ultimate goal is to develop a tool to compute homology in a completely combinatorial way.

§5.3.1 Cellular Complex

We now want to develop a tool to compute homology of arbitrary CW complex. We do that inductively, so it suffices to determine the relation between the $H_q(\operatorname{Sk}_{n-1} X)$ and $H_q(\operatorname{Sk}_n X)$. First, we have the long exact sequence.

$$\ldots \longrightarrow H_q(\operatorname{Sk}_{n-1} X) \longrightarrow H_q(\operatorname{Sk}_n X) \longrightarrow H_q(\operatorname{Sk}_n X, \operatorname{Sk}_{n-1} X) \xrightarrow{\delta} H_{q-1}(\operatorname{Sk}_{n-1} X) \longrightarrow \ldots. \tag{5.1}$$

Assuming $q > 0$, then

$$H_q(\operatorname{Sk}_n X, \operatorname{Sk}_{n-1} X) \simeq H_q(\operatorname{Sk}_n X / \operatorname{Sk}_{n-1} X) \sim H_q\left(\bigvee_{i\in I} S^n\right)$$

for some index set I corresponding to the n-cells.

Thus, we want to understand the homology group of a wedge. We have the exact sequence of topological space

$$\coprod_{i\in I_n}\{\text{point}\} \hookrightarrow \coprod_{i\in I_n} S^n \longrightarrow \bigvee_{i\in I_n} S^n,$$

which induces the long exact sequence of homology groups

$$H_q\left(\coprod_{i\in I_n}\{\text{point}\}\right) \longrightarrow H_q\left(\coprod_{i\in I_n} S^n\right) \longrightarrow H_q\left(\bigvee_{i\in I} S^n\right) \longrightarrow H_{q-1}\left(\coprod_{i\in I_n}\{\text{point}\}\right).$$

We analyze a couple cases.

- If $q = 0$, then $H_q\left(\bigvee_{i\in I_n} S^n\right) = \mathbb{Z}$.
- If $q > 1$, then the first and fourth factor are 0, so the second and third factors are isomorphic. The second factor is 0 if $q \neq n$ and $\bigoplus_{i\in I_n}\mathbb{Z}$ if $q = n$.
- If $q = 1$, then we note that the map from

  $$H_0\left(\coprod_{i\in I_n}\{\text{point}\}\right) \longrightarrow H_0\left(\coprod_{i\in I_n} S^n\right)$$

  is has trivial kernel, so the map from $H_1\left(\bigvee_{i\in I} S^n\right) \to H_0\left(\coprod_{i\in I_n}\{\text{point}\}\right)$ is zero map. Thus, we still have that the second and third factors are isomorphic.

Collating all the cases gives

$$H_q\left(\bigvee_{i\in I_n} S^n\right) = \begin{cases} 0 & q \neq 0, n \\ \bigoplus_{i\in I_n}\mathbb{Z} & q = n \\ \mathbb{Z} & q = 0 \end{cases}. \tag{5.2}$$

The upshot of (5.2) is that the long exact sequence in (5.1) implies

$$H_q(\operatorname{Sk}_{n-1} X) = H_q(\operatorname{Sk}_n X) \text{ unless } q = n-1 \text{ or } n.$$

In particular, we now deduce that

Corollary 5.3.1.

Let X be a CW Complex. Then,

(a) $H_q(\mathrm{Sk}_n X) = 0$ for all $q > n$.

(b) $H_q(\mathrm{Sk}_n X) = H_q(X)$ for all $q < n$.

Now, we want to understand the case of $q = n - 1$ or n more. To do that, we make the following definition

Definition 5.3.2 (Cellular chain complex).

Suppose X is a CW complex. Define, for each integer $n \geq 0$, the **cellular chain complex**

$$C_n^{\text{cell}} = H_n(\mathrm{Sk}_n X, \mathrm{Sk}_{n-1} X) \overset{\text{if } n \neq 0}{=} \bigoplus_{i \in I_n} \mathbb{Z}.$$

We denote by d, the composition

$$\underbrace{H_n(\mathrm{Sk}_n X, \mathrm{Sk}_{n-1} X)}_{=C_n^{\text{cell}}(x)} \xrightarrow{\delta} H_{n-1}(\mathrm{Sk}_{n-1} X) \longrightarrow \underbrace{H_{n-1}(\mathrm{Sk}_{n-1} X, \mathrm{Sk}_{n-2} X)}_{=C_{n-1}^{\text{cell}}(X)}.$$

We now prove the following theorem, which will allow us to reduce the chain complex of singular homology (which has uncountably many generator) to the chain complex of cells (which often has finitely many generator).

Theorem 5.3.3 (Cellular chain complex gives the same homology group).

$(C_n^{\text{cell}}(X), d)$ forms a chain complex with homology groups isomorphic to X.

▶ *Proof.* By long exact sequence on $\mathrm{Sk}_{n-1}(X) \subset \mathrm{Sk}_n(X)$, we have the long exact sequence

$$\begin{aligned} &H_n(\mathrm{Sk}_{n-1}(X)) \longrightarrow H_n(\mathrm{Sk}_n X) \longrightarrow C_n^{\text{cell}}(X) \\ &\quad \xrightarrow{\delta} H_{n-1}(\mathrm{Sk}_{n-1} X) \longrightarrow H_{n-1}(\mathrm{Sk}_n X) \longrightarrow H_{n-1}(\mathrm{Sk}_n X, \mathrm{Sk}_{n-1} X). \end{aligned}$$

However, Corollary 5.3.1 implies that $H_n(\mathrm{Sk}_{n-1}(X)) = 0$ and $H_{n-1}(\mathrm{Sk}_n X) = H_{n-1}(X)$. Moreover, we also have $H_{n-1}(\mathrm{Sk}_n X, \mathrm{Sk}_{n-1} X) = 0$. Thus, the above long exact sequence simplifies to

$$0 \longrightarrow H_n(\mathrm{Sk}_n X) \longrightarrow C_n^{\text{cell}}(X) \xrightarrow{\delta} H_{n-1}(\mathrm{Sk}_{n-1} X) \longrightarrow H_{n-1}(X) \longrightarrow 0. \tag{5.3}$$

We examine the following diagram, where all rows and columns (not the diagonal) are exact from (5.3) applied on $n+1$, n, and $n-1$.

$$\begin{tikzcd}
 & 0 \arrow[d] & & & \\
 & H_{n+1}(\operatorname{Sk}_{n+1} X) \arrow[d] & & & \\
 & C^{\text{cell}}_{n+1}(X) \arrow[d, "\delta_1"] \arrow[dr, "d_1"] & & 0 \arrow[d] & \\
0 \arrow[r] & H_n(\operatorname{Sk}_n X) \arrow[r, "f"] \arrow[d] & C^{\text{cell}}_n(X) \arrow[r, "\delta_2"] \arrow[dr, "d_2"] & H_{n-1}(\operatorname{Sk}_{n-1} X) \arrow[r] \arrow[d, "j"] & \dots \\
 & H_n(X) \arrow[d] & & C^{\text{cell}}_{n-1}(X) & \\
 & 0 & & &
\end{tikzcd}$$

We first show that $C^{\text{cell}}_\bullet(X)$ is indeed a chain complex. This amounts to showing that $d_2 d_1 = 0$ in diagram above. The composition $d_2 d_1$ is $j\delta_2 f\delta_1$. However, $\delta_2 f = 0$ by exactness at $C^{\text{cell}}_n(X)$, which implies that $d_2 d_1 = 0$.

Next we show the homology. Note that f and j are injective by exactness at $H_n(\operatorname{Sk}_n X)$ and $H_{n-1}(\operatorname{Sk}_{n-1} X)$. Therefore,

$$\begin{aligned}
\operatorname{Im}(d_1) &= \operatorname{Im}(f\delta_1) = \operatorname{Im}(\delta_1) \\
\operatorname{Ker}(d_2) &= \operatorname{Ker}(j\delta_2) = \operatorname{Ker}(\delta_2) = \operatorname{Im}(f) \simeq H_n(\operatorname{Sk}_n X)
\end{aligned}$$

Thus,

$$H_n(C^{\text{cell}}_n(X)) = \frac{\operatorname{Ker}(d_2)}{\operatorname{Im}(d_1)} \simeq \frac{H_n(\operatorname{Sk}_n X)}{\operatorname{Im}\delta_1} = \frac{H_n(\operatorname{Sk}_n X)}{\operatorname{Ker}(H_n(\operatorname{Sk}_n X) \to H_n(X))} \simeq H_n(X) \qquad \square$$

§5.3.2 Computing Boundary Map

Theorem 5.3.3 allows us to work with chain complex of finitely-generated abelian groups. The only thing left to do is to figure out how to compute the boundary map $\partial_n : C^{\text{cell}}_n(X) \to C^{\text{cell}}_{n-1}(X)$.

When $n = 1$, this map is simply the boundary map. In other words, the boundary of the cell that is a segment from x to y is simply $y - x$.

For general $n > 1$, we need the notion of degree (Definition 5.2.4). The recipe for computing boundary is as follows:

Theorem 5.3.4 (Cellular Boundary Formula)**.**

Let $n \geq 2$ be an integer, and let D be an n-cell of a CW complex X. Suppose that $\{E_i : i \in I\}$ are all the $(n-1)$-cells of X. Then,

$$d_n(D) = \sum_i e_i E_i,$$

where e_i denote the degree of the composition

$$S^{n-1} = \partial D \xrightarrow{\text{attach}} \operatorname{Sk}_{n-1} X \twoheadrightarrow E_i = S^{n-1},$$

where the second arrow is collapsing $\operatorname{Sk}_{n-1} X \setminus E_i$ to a point.

▶ *Proof.* Let the map in question be Δ_i. We break it the second quotient step down to two steps, so Δ_i is the composition of three maps

$$\partial D \longrightarrow \operatorname{Sk}_{n-1} X \longrightarrow \operatorname{Sk}_{n-1} X / \operatorname{Sk}_{n-2} X \longrightarrow E_i,$$

We take homology on all the maps and noting that by excision (Theorem 4.4.2),

$$H_{n-1}(\operatorname{Sk}_{n-1} X/\operatorname{Sk}_{n-2} X) = H_{n-1}(\operatorname{Sk}_{n-1} X, \operatorname{Sk}_{n-2} X) = C^{\text{cell}}_{n-1}(X),$$

so we have the chain of mapping

$$\begin{tikzcd}
H_{n-1}(\partial D) \arrow[r, "\cdot d_i"] \arrow[d] & H_{n-1}(E_i) \\
H_{n-1}(\operatorname{Sk}_{n-1} X) \arrow[r] & C^{\text{cell}}_{n-1}(X) \arrow[u]
\end{tikzcd} \tag{5.4}$$

Next, the commuting square

$$\begin{tikzcd}
\partial D \arrow[r, hook] \arrow[d] & D \arrow[d] \\
\operatorname{Sk}_{n-1} X \arrow[r, hook] & \operatorname{Sk}_n X
\end{tikzcd}$$

induces a commuting square which is part of the long exact sequence

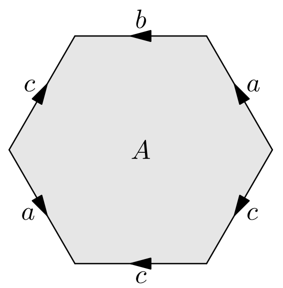
$$\begin{tikzcd}
0 = H_n(D) \arrow[r] & H_n(D, \partial D) \arrow[r, "\sim"] \arrow[d] & H_{n-1}(\partial D) \arrow[r] \arrow[d] & H_{n-1}(D) = 0 \\
& C^{\text{cell}}_n(X) \arrow[r] & H_{n-1}(\operatorname{Sk}_{n-1} X), &
\end{tikzcd} \tag{5.5}$$

so combining (5.4) and (5.5), we get the diagram

$$\begin{tikzcd}
H_n(D, \partial D) \arrow[r, "\sim"] \arrow[d] \arrow[rr, bend left, "\cdot e_i"] & H_{n-1}(\partial D) \arrow[r, "\cdot e_i"] \arrow[d] & H_{n-1}(E_i) \\
C^{\text{cell}}_n(X) \arrow[r] \arrow[rr, bend right, "d_n"'] & H_{n-1}(\operatorname{Sk}_{n-1} X) \arrow[r] & C^{\text{cell}}_{n-1}(X) \arrow[u]
\end{tikzcd}$$

Now, suppose that we start from D in the top left corner, which gets send to D in the bottom left corner. Thus, the bottom right corner is $d_n D$, which is a linear combination of a number of $(n-1)$-cells. However, upon projection to $H_{n-1}(E_i)$, all cells vanish except the cell E_i. Meanwhile, the top arrow force the map from $H_n(D, \partial D) \to H_{n-1}(E_i)$ to be multiplication by e_i. Hence, we deduce that the coefficient in front of E_i must be d_i. □

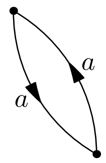
For concreteness, let us write down the recipe for $n = 2$. A natural thing to do is to follow along the cell in counterclockwise direction. If an edge is in the same direction as the direct we are traveling, we add it. If an edge is against the direction we are traveling, we subtract it. For example, consider the following cell:



If we start from the rightmost vertex and travel counterclockwise, we will get edges a, b, $-c$, a, $-c$, $-c$ in that order, so boundary of A is

$$\partial A = a + b + c + a + (-c) + (-c) = 2a + b - c.$$

Why does this match with the formula in Theorem 5.3.4? Suppose we want to determine the coefficient of edge a. Then, we would need to collapse everything else to a single point, leaving with the diagram



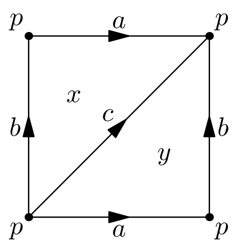
In reality, those two edges should be the same, so the map would be more like winding around the circle twice, hence it has degree 2, and so the coefficient in front of a is indeed 2.

§5.4 Examples

Armed with the new theory, we now demonstrate how to compute homology groups of arbitrary CW complex.

§5.4.1 Homology of a torus.

A torus can be depicted as a CW complex as follows.



There are one 0-cell $\{p\}$, three 1-cells $\{a, b, c\}$, and two 2-cells $\{x, y\}$.

Then, $C_*^{\text{cell}}(X)$ is a chain complex

$$0 \longrightarrow \mathbb{Z}\{x, y\} \xrightarrow{d_2} \mathbb{Z}\{a, b, c\} \xrightarrow{d_1} \mathbb{Z}\{p\} \longrightarrow 0,$$

where the two boundary maps are by

$$d_2 x = a - c + b, \quad d_2 y = b - c + a$$

and

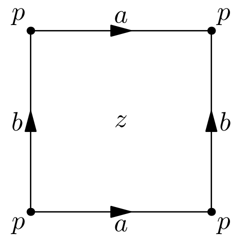
$$d_1 a = d_1 b = d_1 c = p - p = 0.$$

Computing homology now is extremely simple. We have

$$\begin{aligned} H_1(X) &= \frac{\operatorname{Ker}(d_1)}{\operatorname{Im}(d_2)} = \frac{\mathbb{Z}\{a, b, c\}}{(a + b - c)} = \mathbb{Z} \oplus \mathbb{Z} \\ H_2(X) &= \frac{\operatorname{Ker}(d_2)}{\operatorname{Im}(d_3)} = \frac{\mathbb{Z}\{x - y\}}{(0)} = \mathbb{Z} \\ H_n(X) &= 0 \qquad \text{for all } n \geq 3. \end{aligned}$$

Notice that in this case, the CW complex is the same as simplicial complex, so in fact, this proves that homology of a topological space is the same as homology of semisimplicial sets (if we *cut* things).

On the other hand, the CW complex theory allows us to use cells that's not a triangle. Another cell structure that can be used to compute homology is



Then, we instead of a chain complex $C_*^{\text{cell}}(X)$

$$\mathbb{Z}\{z\} \xrightarrow{d_2} \mathbb{Z}\{a,b\} \xrightarrow{d_1} \mathbb{Z}\{p\} \longrightarrow 0,$$

and by Theorem 5.3.4, the boundary map is

$$d_1(a) = d_1(b) = 0, \qquad d_2(z) = a - b - a + b = 0,$$

so we have

$$H_1(X) = \frac{\operatorname{Ker}(d_1)}{\operatorname{Im}(d_2)} = \frac{\mathbb{Z}\{a,b\}}{0} \simeq \mathbb{Z} \oplus \mathbb{Z}.$$

In conclusion, we deduce the following.

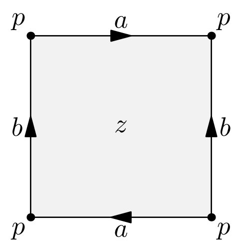
Proposition 5.4.1 (Homology of torus)**.**

If $X = S^1 \times S^1$ is the torus, then

$$H_n(X) = \begin{cases} \mathbb{Z} & n = 0 \\ \mathbb{Z} \oplus \mathbb{Z} & n = 1 \\ \mathbb{Z} & n = 2 \\ 0 & n \geq 3 \end{cases}.$$

§5.4.2 Klein Bottle

Computing homology of a Klein bottle is now easy. Recall that a Klein bottle K is a quotient of a square, where we identify edge as follows:



It's easy to see that all four corners are identified as the same point. Thus, a cell structure of K has 1 0-cell, 2 1-cells, and 1 2-cells, so a cellular chain complex looks like

$$\mathbb{Z}\{z\} \xrightarrow{d_2} \mathbb{Z}\{a,b\} \xrightarrow{d_1} \mathbb{Z}\{p\} \longrightarrow 0,$$

and the boundary maps are

$$d_1 a = d_1 b = p - p = 0.$$

$$d_2 z = (-a) + b + (-a) + (-b) = -2a,$$

and finally, we can compute the homology

$$\begin{aligned} H_1(X) &= \frac{\operatorname{Ker}(d_1)}{\operatorname{Im}(d_2)} = \frac{\mathbb{Z}\{a,b\}}{(2a)} = \mathbb{Z}\oplus\mathbb{Z}/2, \\ H_2(X) &= \frac{\operatorname{Ker}(d_2)}{\operatorname{Im}(d_3)} = 0, \\ H_n(X) &= 0 \qquad \text{for all } n \geq 3. \end{aligned}$$

Hence, we deduce that

Proposition 5.4.2 (Homology of Klein bottle)**.**

If K is the Klein bottle, then

$$H_n(K) = \begin{cases} \mathbb{Z} & n = 0 \\ \mathbb{Z}\oplus\mathbb{Z}/2 & n = 1 \\ 0 & n \geq 2 \end{cases}.$$

§5.4.3 Real Projective Space

We now compute the homology group of $\mathbb{RP}^n$.

Recall that the cell structure of $\mathbb{RP}^n$ is given as follows: $\mathbb{RP}^n$ is obtained by attaching D^n onto $\mathbb{RP}^{n-1}$, where the attaching map is the quotient projection. This gives the CW structure of $\mathbb{RP}^n$ by induction, where we have $\mathrm{Sk}_{n-1}\mathbb{RP}^n = \mathbb{RP}^{n-1}$. There is one cell for each dimension from 0, 1, 2, $\ldots$, n. Therefore, the cellular chain complex looks like

$$C_*^{\text{cell}}(\mathbb{RP}^n) : \mathbb{Z}\{c_n\} \longrightarrow \mathbb{Z}\{c_{n-1}\} \longrightarrow \ldots \longrightarrow \mathbb{Z}\{c_1\} \longrightarrow \mathbb{Z}\{c_0\}$$

where $c_0, c_1, \ldots, c_n$ denote the cell of dimension $0, 1, \ldots, n$, respectively.

Exercise 5.4.3. Before you read what follows, compute the homology group of $\mathbb{RP}^2$ using the method in previous subsections. Can you do $\mathbb{RP}^3$?

We now compute the boundary map using Cellular Boundary Formula (Theorem 5.3.4). To do this, we need to compute the degree of the composition

$$S^{k-1} \xrightarrow{\text{quotient}} \mathbb{RP}^{k-1} \longrightarrow \mathbb{RP}^{k-1}/\mathbb{RP}^{k-2} = S^{k-1}.$$

To do this, we investigate what the right map is doing on the cover S^{k-1} of $\mathbb{RP}^{k-1}$, which projects to the cover $S^{k-1}/S^{k-2} \simeq S^{k-1} \vee S^{k-1}$ of $\mathbb{RP}^{k-1}/\mathbb{RP}^{k-2}$ We have the commutative diagram

$$\begin{tikzcd} \mathbb{RP}^{k-1} \arrow[r] & \mathbb{RP}^{k-1}/\mathbb{RP}^{k-2} \\ S^{k-1} \arrow[u] \arrow[ur, "\deg=?"] \arrow[r] & S^{k-1} \vee S^{k-1} \arrow[u] \end{tikzcd}.$$

The lower triangle give rise to the map between homology groups

$$\begin{tikzcd} & \overbrace{H_{k-1}(\mathbb{RP}^{k-1}/\mathbb{RP}^{k-2})}^{\mathbb{Z}} \\ \underbrace{H_{k-1}(S^{k-1})}_{\mathbb{Z}} \arrow[ur] \arrow[r] & \underbrace{H_{k-1}(S^{k-1}\vee S^{k-1})}_{\mathbb{Z}\oplus\mathbb{Z}} \arrow[u] \end{tikzcd}.$$

The right arrow map maps S^{k-1} to itself on both wedges Therefore, it sends 1 to $(1,1)$. However, the up arrow maps the first wedge to itself but the second wedge to its antipode. Therefore, it sends $(1,0)$ to 1 and $(0,1)$ to $(-1)^k$. Therefore, the overall composition is

$$\begin{tikzcd} & 1+(-1)^k \\ 1 \arrow[ur, mapsto] \arrow[r, mapsto] & (1,1) \arrow[u, mapsto] \end{tikzcd}.$$

Hence, we deduce that the boundary map $\mathbb{Z}\{c_k\} \to \mathbb{Z}\{c_{k-1}\}$ sends c_k to $(1+(-1)^k)c_{n-1}$, which is 0 if k is odd and $2c_{k-1}$ if k is even. In particular, the chain complex looks like

$$C_*^{\text{cell}}(\mathbb{RP}^n) : \mathbb{Z}\{c_n\} \xrightarrow{2 \text{ or } 0} \dots \xrightarrow{\cdot 0} \mathbb{Z}\{c_2\} \xrightarrow{\cdot 2} \mathbb{Z}\{c_1\} \xrightarrow{\cdot 0} \mathbb{Z}\{c_0\},$$

and we can easily read off the homology group as

Theorem 5.4.4.

The homology groups of $\mathbb{RP}^n$ are

$$H_k(\mathbb{RP}^n) = \begin{cases} \mathbb{Z} & k = 0 \\ \mathbb{Z} & k = n \text{ are odd} \\ \mathbb{Z}/2 & k \text{ odd}, 0 < k < n \\ 0 & \text{otherwise.} \end{cases}$$

For convenience, here is the table for small k and n. Missing entries are 0's.

	$\mathbb{RP}^1$	$\mathbb{RP}^2$	$\mathbb{RP}^3$	$\mathbb{RP}^4$	$\mathbb{RP}^5$	$\mathbb{RP}^6$
H_0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
H_1	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$
H_2		0	0	0	0	0
H_3			$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$
H_4				0	0	0
H_5					$\mathbb{Z}$	$\mathbb{Z}/2$
H_6						0

§5.5 Euler Characteristic

Euler characteristic is a topological invariant that is much easier to compute than homology.

Definition 5.5.1 (Euler characteristic)**.**

If X is a finite CW-complex, define the **Euler characteristic** of X as

$$\chi(X) = \sum_{n \in \mathbb{Z}} (-1)^n \,(\#\, n\text{-cells in } X)$$

Example 5.5.2 (S^2, torus, and $\mathbb{RP}^2$)**.**

S^2 has a CW structure with a 0-cell and a 2-cell, so $\chi(S^2) = 1 - 0 + 1 = 2$.

Another CW structure of S^2 that we discussed has two 0-cells, two 1-cells, and two 2-cells. Thus, $\chi(S^2) = 2 - 2 + 2 = 2$.

The CW structure of torus given in Example 1.2.16 contains one 0-cell, two 1-cells, and one 2-cell. Thus, $\chi(\text{torus}) = 1 - 2 + 1 = 0$.

The CW structure of $\mathbb{RP}^2$ given in Section 5.4.3 has 1 0-cell, 1 1-cell, and 1 2-cell, so we have $\chi(\mathbb{RP}^2) = 1 - 1 + 1 = 1$.

Let us now explain how to compute the Euler characteristic from homology group. First, define the **rank** of a finitely-generated abelian group.

Definition 5.5.3 (Rank of an abelian group).

The **rank** of a finitely-generated abelian group A is the number r such that $A \simeq \mathbb{Z}^r \oplus B$, where B finite.

We have the following fact from algebra, whose proof is omitted.

Lemma 5.5.4.

Let
$$0 \to A_1 \to A_2 \to A_3 \to 0$$
be an exact sequence of abelian groups, then
$$\operatorname{rank}(A_1) + \operatorname{rank}(A_3) = \operatorname{rank}(A_2).$$

Theorem 5.5.5 (Euler characteristic and homology).

If X is a finite CW-complex, then
$$\chi(X) = \sum_{k=0}^{\infty} (-1)^k \operatorname{rank}(H_k(X)).$$

Notice that $H_k(X)$ is finitely-generated since it is computed from $C_*^{\text{cell}}(X)$, which is a finitely-generated complex. By the same reason, the sum is finite.

▶ *Proof.* Consider the exact sequence
$$0 \longrightarrow Z_k(X) \longrightarrow C_k^{\text{cell}}(X) \longrightarrow B_{k-1}(X) \longrightarrow 0,$$
which is exact because $Z_k(X)$ and $B_{k-1}(X)$ are kernels and images. Thus, we have that
$$\#(k\text{-cells}) = \operatorname{rank} Z_k(X) + \operatorname{rank} B_{k-1}(X).$$
Now, we also have this exact sequence
$$0 \longrightarrow B_k \longrightarrow Z_k \longrightarrow H_k \longrightarrow 0,$$
so we have that $\operatorname{rank} Z_k(X) = \operatorname{rank} B_k(X) + \operatorname{rank} H_k(X)$. Using these two equations together, we find that
$$\begin{aligned}
\chi(X) &= \sum_{k=0}^{\infty} (-1)^k \#(k\text{-cells}) \\
&= \sum_{k=0}^{\infty} (-1)^k (\operatorname{rank} B_k(X) + \operatorname{rank} H_k(X) + \operatorname{rank} B_{k-1}(X)) \\
&= \sum_{k=0}^{\infty} (-1)^k \operatorname{rank} H_k(X)
\end{aligned}$$ □

In particular, this implies that $\chi(X)$ does not depend on the cell structure of X and also is a invariant that is extremely easy to compute.

§5.6 Problems

Problem 5.A (Homology of Wedge Sum)**.** Let X and Y be a topological space. Let $x_0 \in X$ and $y_0 \in Y$ such that there exists a contractible open neighborhood $x_0 \in U$ and $y_0 \in V$. Prove that

$$H_n(X \vee Y) = H_n(X) \oplus H_n(Y)$$

for all $n \geq 1$.

Problem 5.B. Prove that any non-surjective map $S^n \to S^n$ must have degree 0.

Problem 5.C. Compute the homology groups $H_k(X)$ for any k, where X is the following spaces.

(a) X is $\mathbb{R}^n$ with p points removed.

(b) $X = \Sigma_g$, the g-holed torus.

(c) $X = \mathbb{CP}^n$, the complex projective space.

(d) X is a quotient of S^n, identifying the north pole and the south pole to the same point.

(e) $X = \mathrm{Gr}_2(\mathbb{C}^4)$, the space parametrizing 2-dimensional $\mathbb{C}$-vector space of $\mathbb{C}^4$ (hint: the cell structure is given by row-reduced echelon form).

(f) $X = S^m \times S^n$ (this will be trivialized by Theorem 6.6.1 in the next chapter, but you should do it using cellular homology for now).

(g) $X = (S^n \times S^n) \setminus \Delta$, where $\Delta = \{(x, x) : x \in S^n\}$ is the diagonal.

(h) X is **Lens space**, defined as the quotient of $S^3 \subseteq \mathbb{C}^2$ by the action of μ_p (the group of p-th root of unity) by $\zeta \cdot (z_1, z_2) = (\zeta z_1, \zeta z_2)$.

Problem 5.D. Let X and Y be finite CW-complexes. Prove that $\chi(X \times Y) = \chi(X) \cdot \chi(Y)$.

Problem 5.E. Let X be a topological spaces, and let U and V be open sets such that $X = U \cup V$. Suppose that all homology groups of X, U, V, and $U \cap V$ are finitely-generated. Prove that

$$\chi(X) = \chi(U) + \chi(V) - \chi(U \cap V).$$

6 Homology with Coefficients

In this chapter, we will discuss homology, where in place of $\mathbb{Z}$, the coefficients can be element from any commutative ring R. In particular, the singular chain complex will consists of R-modules (instead of abelian groups), and so homology $H_n(X, R)$ will be R-modules instead.

In the case where $R = k$ is a field, $H_n(X, k)$ will be a k-vector space, and thus be of the form $k^{\oplus r}$ for some integer r. This makes homology over k convenient to represent and compute. In fact, we will quickly see that there are spaces where we can compute homology over some fields k, but not homology over $\mathbb{Z}$.

$\boxed{\text{For the rest of the book, let } R \text{ denote a commutative ring and } k \text{ denote a field.}}$

Rings that are of interest in algebraic topology are simple, such as $\mathbb{Q}$, $\mathbb{Z}$, and $\mathbb{F}_p$. For most of the time, they are fields or a PID.

After a quick setup in Section 6.1, the rest of this chapters explore the relation between homology with coefficients in $\mathbb{Z}$ and with coefficients in commutative ring R, culminating with a key result that **for any ring R, $H_n(X, R)$ is uniquely determined from $H_n(X, \mathbb{Z})$**. Doing this requires tools from homological algebra that allows us to *multiply* chain complexes, so we take a detour to develop those tools. The payoffs are the following:

- a formula to compute $H_n(X, R)$ from $H_n(X, \mathbb{Z})$ for any ring R (Corollary 6.5.2);
- a formula to compute $H_n(X \times Y, \mathbb{Z})$ from $H_n(X, \mathbb{Z})$ and $H_n(Y, \mathbb{Z})$ (Corollary 6.6.9); and
- a new invariant induced from the diagonal map $X \to X \times X$, which we develop it further in the next chapter.

§6.1 Definition

The setup is exactly the same as the usual homology group: one simply replace an abelian group with an R-module.

Definition 6.1.1 (Homology with Coefficients).

Given a semisimplicial set X_*, we define a chain complex $S_*(X, R)$ of R-modules, where $S_n(X, R)$ is the free R-modules generated by X_n.

The homology groups of $S_*(X, R)$ are defined in exactly the same way as when $R = \mathbb{Z}$ and is denoted $H_k(X, R)$. The relative homology $H_k((X, A), R)$ is also defined in the same way.

When $R = \mathbb{Z}$, we recover the usual chain complex $S_*(X, \mathbb{Z}) \simeq S_*(X)$ and $H_k(X, \mathbb{Z}) \simeq H_*(X)$.

Everything we have done so far works on arbitrary commutative ring R. In particular, $H_k(\bullet, R)$ satisfies all properties of homology over $\mathbb{Z}$.

- The homology of a point is given by

$$H_k(\text{point}, R) = \begin{cases} R & k = 0 \\ 0 & k \neq 0 \end{cases}.$$

- Homology with R-coefficient is homotopy invariant: if f and g are homotopic, then $H_n(f, R) = H_n(g, R)$.

- Excision still works. In particular, if the closure of U is contained in interior of A, then $H_n((X, A), R) \simeq H_n((X - U, A - U), R)$.

- Cellular chain complex also works fine.

Example 6.1.2 ($\mathbb{RP}^2$)**.**

For $\mathbb{Z}$, we have the chain complex

$$C_*^{\text{cell}}(\mathbb{RP}^2, \mathbb{Z}) : \ldots \longrightarrow 0 \longrightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \longrightarrow 0.$$

Similarly, in $\mathbb{Q}$, we have the chain complex

$$C_*^{\text{cell}}(\mathbb{RP}^2, \mathbb{Q}) : \ldots \longrightarrow 0 \longrightarrow \mathbb{Q} \xrightarrow{2} \mathbb{Q} \xrightarrow{0} \mathbb{Q} \longrightarrow 0.$$

Notice that in $\mathbb{Q}$, multiplication by 2 is bijective. This allow us to compute

$$H_k(\mathbb{RP}^2, \mathbb{Q}) = \begin{cases} \mathbb{Q} & k = 0 \\ 0 & k \neq 0 \end{cases}.$$

Notice that here, we do not get the 2-torsion that allows us to distinguish between $\mathbb{RP}^2$ and a point.

On the other hand, if we compute homology with $\mathbb{F}_2$ coefficients, we will get the chain complex

$$C_*^{\text{cell}}(\mathbb{RP}^2, \mathbb{F}_2) : \ldots \longrightarrow 0 \longrightarrow \mathbb{F}_2 \xrightarrow{2} \mathbb{F}_2 \xrightarrow{0} \mathbb{F}_2 \longrightarrow 0.$$

In fact, all maps turn out to be zero map. We get that

$$H_k(\mathbb{RP}^2, \mathbb{F}_2) = \begin{cases} \mathbb{F}_2 & k = 0, 1, 2 \\ 0 & \text{otherwise.} \end{cases}$$

Exercise 6.1.3 (to be used in the next section)**.** Prove that

$$H_k(S^n, \mathbb{F}_2) = \begin{cases} \mathbb{F}_2 & k = 0, n \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad H_k(\mathbb{RP}^n, \mathbb{F}_2) = \begin{cases} \mathbb{F}_2 & k = 0, 1, 2, \ldots, n \\ 0 & \text{otherwise.} \end{cases}$$

Corollary 6.5.2 shows that $H_*(X, R)$ is uniquely determined from $H_*(X, \mathbb{Z})$. However, as seen from the example above, $H_*(X, R)$ loses some information; it cannot be used to recover $H_*(X, \mathbb{Z})$. In some sense, $H_*(X, \mathbb{Z})$ conveys the most information out of all possible coefficients.

Why do we care about homology with arbitrary coefficients at all when it's all defined by homology with $\mathbb{Z}$? One possible answer is that there are some spaces that we only understand homology at some particular ring.

Example 6.1.4.

For any topological space X, consider the **unordered configuration space**

$$C_k(X) = \frac{\{(x_1, \ldots, x_k) : x_i \in X, x_i \neq x_j \text{ if } i \neq j\}}{S_k},$$

where S_k acts by permuting points in the k-tuple. In other words, $C_k(X)$ is a space of subsets of size k of X.

It is an open problem to compute $H_n(C_k(T), \mathbb{Z})$, where T is a torus. We know a partial answer.

- $H_n(C_k(T), \mathbb{F}_2)$ is known from [ML88].
- $H_n(C_k(T), \mathbb{Q})$ is known from [FT00].

However, other rings are much harder. There are some active research to compute homology of $H_n(C_k(T), \mathbb{F}_p)$: [BHK24] and [CZ24].

§6.2 Interlude: Borsuk Ulam Theorem

As an application of what we have so far, we will prove Borsuk-Ulam Theorem. This theorem is special in that the proof will showcase how homology with coefficients helps us see the space in a different way.

Theorem 6.2.1 (Borsuk-Ulam Theorem)**.**

For every continuous map $g : S^n \to \mathbb{R}^n$, there exists two antipodal points x and $-x$ such that $g(x) = g(-x)$.

The case $n = 2$ of this theorem is often known as the "meteorologist theorem": at any time, there are two antipodal points on the earth that have the same temperature and pressure. Problem 6.A showcases a neat application of this theorem in combinatorics.

In order to prove this, we prove the following key lemma.

Lemma 6.2.2 (Odd map has odd degree)**.**

Let $f : S^n \to S^n$ be an odd map (i.e., $f(-x) = -f(x)$ for all $x \in S^n$). Then, f has odd degree.

We first explain how to deduce Borsuk-Ulam Theorem (Theorem 6.2.1) from Lemma 6.2.2.

► *Proof of Theorem 6.2.1 assuming Lemma 6.2.2.* Let $g : S^n \to \mathbb{R}^n$ be a continuous map. Consider the map

$$\begin{aligned} f : S^n &\to \mathbb{R}^n \\ x &\mapsto g(x) - g(-x). \end{aligned}$$

Clearly, f is odd. Assume for the contradiction that $f(x) \neq 0$ for all x. Thus, we get the induced map

$$\begin{aligned} \widetilde{f} : S^n &\to S^{n-1} \\ x &\mapsto \tfrac{f(x)}{|f(x)|}. \end{aligned}$$

The restriction of this map onto the equator is the odd map $S^{n-1} \to S^{n-1}$, so by Lemma 6.2.2, it has odd degree, hence not equal to 0.

However, $\widetilde{f}|_{S_{n-1}}$ factors to $S^{n-1} \hookrightarrow S^n \to S^{n-1}$, inducing the morphism of homology groups $H_{n-1}(S^{n-1}) \to H_{n-1}(S^n) \to H_{n-1}(S^{n-1})$. However, the middle factor is 0, so $\widetilde{f}|_{S_{n-1}}$ must have degree 0, a contradiction. □

For the rest of this section, we prove Lemma 6.2.2.

First, we note the following two observations.

- A map $f : S^n \to S^n$ has odd degree iff

$$f_* : H_n(S^n, \mathbb{F}_2) \to H_n(S^n, \mathbb{F}_2)$$

  is an isomorphism. Note that both of these are $\mathbb{F}_2$ by Exercise 6.1.3. Thus, we can reduce to studying homology in $\mathbb{F}_2$.

- We have a double cover $S^n \to \mathbb{RP}^n$, and an odd map passes to the map on quotient

$$\begin{tikzcd} S^n \arrow[r] \arrow[d, "f"] & \mathbb{RP}^n \arrow[d, "\overline{f}"] \\ S^n \arrow[r] & \mathbb{RP}^n \end{tikzcd} .$$

Recall that a **double cover** of a connected topological space X is a covering space $p : \widetilde{X} \to X$ such that $|p^{-1}(x)| = 2$ for all $x \in X$.

It turns out that double covers induce a useful structure on chains and homology, as outlined in the following proposition.

Proposition 6.2.3.

Let $p : \widetilde{X} \to X$ be a double cover of connected topological spaces. Then, there exists a short exact sequence of chain complexes

$$0 \longrightarrow S_*(X, \mathbb{F}_2) \xrightarrow{\tau} S_*(\widetilde{X}, \mathbb{F}_2) \xrightarrow{p_*} S_*(X, \mathbb{F}_2) \longrightarrow 0.$$

The map τ is called **transfer map**.

By Theorem 4.3.11, the exact sequence in the above proposition gives the long exact sequence of homology group

$$\begin{tikzcd} \dots \arrow[r] & H_n(X, \mathbb{F}_2) \arrow[r] & H_n(\widetilde{X}, \mathbb{F}_2) \arrow[r] & H_n(X, \mathbb{F}_2) \arrow[dll] & \\ & H_{n-1}(X, \mathbb{F}_2) \arrow[r] & H_{n-1}(\widetilde{X}, \mathbb{F}_2) \arrow[r] & H_{n-1}(X, \mathbb{F}_2) \arrow[r] & \dots \end{tikzcd}$$

▶ *Proof of Proposition 6.2.3.* First, we show that p_* is surjective. This is equivalent to that any map $\sigma : \Delta^n \to X$ lifts to a map $\sigma : \Delta^n \to \widetilde{X}$, which follows from the general lifting criterion Lemma 3.2.12 (since Δ^n is simply connected). In fact, since Δ^n is connected, Δ^n has exactly two lifts, determined by the image of one point. Let those lifts be $\widetilde{\sigma}_1$ and $\widetilde{\sigma}_2$. Thus, the kernel of p_* is generated by

$$\widetilde{\sigma}_1 + \widetilde{\sigma}_2 \text{ for each } \sigma : \Delta^n \to X.$$

(Remember that we are working on $\mathbb{F}_2$, so plus or minus does not matter.)

This motivates us to define the transfer map by

$$\begin{aligned} \tau : S_*(X, \mathbb{F}_2) &\to S_*(\widetilde{X}, \mathbb{F}_2) \\ \sigma &\mapsto \widetilde{\sigma}_1 + \widetilde{\sigma}_2, \end{aligned}$$

and one can check that this map is compatible with the boundary maps ∂. □

▶ *Proof of Lemma 6.2.2.* Let $f : S^n \to S^n$ be an odd map, inducing a map $\overline{f} : \mathbb{RP}^n \to \mathbb{RP}^n$. The construction of transfer map in Proposition 6.2.3 is natural. We claim that the following commuting diagram with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & S_*(\mathbb{RP}^n, \mathbb{F}_2) \arrow[r] \arrow[d, "\overline{f}_*"] & S_*(S^n, \mathbb{F}_2) \arrow[r] \arrow[d, "f_*"] & S_*(\mathbb{RP}^n, \mathbb{F}_2) \arrow[r] \arrow[d, "\overline{f}_*"] & 0 \\ 0 \arrow[r] & S_*(\mathbb{RP}^n, \mathbb{F}_2) \arrow[r] & S_*(S^n, \mathbb{F}_2) \arrow[r] & S_*(\mathbb{RP}^n, \mathbb{F}_2) \arrow[r] & 0 \end{tikzcd}.$$

The commutativity of left square follows from the uniqueness of lift. In particular, if $\sigma : \Delta^q \to \mathbb{RP}^n$ and σ_1 and σ_2 are the two lifts to $\Delta^q \to S^n$, then $f \circ \sigma_1$ and $f \circ \sigma_2$ are lifts of $\overline{f} \circ \sigma$. Note that they are different lift because f is odd map. Thus, the left square commutes. The commutativity of the right square follows from the fact that f commutes with p.

This induces a long exact sequence of homology groups

$$\begin{tikzcd} H_k(\mathbb{RP}^n, \mathbb{F}_2) \arrow[r] \arrow[d, "H_k(\overline{f})"] & H_k(S^n, \mathbb{F}_2) \arrow[r] \arrow[d, "H_k(f)"] & H_k(\mathbb{RP}^n, \mathbb{F}_2) \arrow[r] \arrow[d, "H_k(\overline{f})"] & H_{k-1}(\mathbb{RP}^n, \mathbb{F}_2) \arrow[r] \arrow[d, "H_{k-1}(\overline{f})"] & H_{k-1}(S^n, \mathbb{F}_2) \arrow[d, "H_k(f)"] \\ H_k(\mathbb{RP}^n, \mathbb{F}_2) \arrow[r] & H_k(S^n, \mathbb{F}_2) \arrow[r] & H_k(\mathbb{RP}^n, \mathbb{F}_2) \arrow[r] & H_{k-1}(\mathbb{RP}^n, \mathbb{F}_2) \arrow[r] & H_{k-1}(S^n, \mathbb{F}_2) \end{tikzcd}.$$

The key claim is that the vertical maps are all isomorphism for all $k \leq n$. To prove this, note that if $k \leq n$, $H_{k-1}(S^n, \mathbb{F}_2) = 0$, so the right three columns gives

$$\begin{tikzcd} H_k(\mathbb{RP}^n, \mathbb{F}_2) \arrow[r] \arrow[d, "H_k(\overline{f})"] & H_{k-1}(\mathbb{RP}^n, \mathbb{F}_2) \arrow[r] \arrow[d, "H_{k-1}(\overline{f})"] & 0 \\ H_k(\mathbb{RP}^n, \mathbb{F}_2) \arrow[r] & H_{k-1}(\mathbb{RP}^n, \mathbb{F}_2) \arrow[r] & 0 \end{tikzcd}.$$

In particular, the horizontal arrows are surjective morphism between $\mathbb{F}_2$, hence an isomorphism. Thus, by induction starting on $H_0(\overline{f})$, we can show that $H_n(\overline{f})$ is isomorphism. Now, the map $H_n(\mathbb{RP}^n, \mathbb{F}_2) \to H_n(S^n, \mathbb{F}_2)$ is also isomorphism, deducing that $H_n(f)$ is an isomorphism. This proves that f has odd degree. □

§6.3 Tensor Product

After defining homology with coefficients, our main goal is to explain the formula to compute $H_n(X, R)$ from $H_n(X, \mathbb{Z})$ and the formula to compute $H_n(X \times Y, \mathbb{Z})$ from $H_n(X, \mathbb{Z})$ and $H_n(Y, \mathbb{Z})$. For this, we will need to take an algebraic detour in this and in the next section. In this section, we explain how to take the product of chain complexes. Since chain complex is built by R-modules, we will first explain how to take the product of two modules first. The right notion for this purpose is **tensor product**.

§6.3.1 Hom and Bilinear Maps

We first review some basic constructions of module.

▶ The Hom functors.

Definition 6.3.1 ($\mathrm{Hom}_R(M, N)$).

Given two R-modules M and N, define the module

$$\mathrm{Hom}_R(M, N) = \{R\text{-module homomorphisms } M \to N\}$$

with the natural module structure $(f+g)(m) = f(m) + g(m)$ and $(rf)(m) = rf(m)$ for any $f, g \in \mathrm{Hom}_R(M, N)$, $m \in M$, and $r \in R$.

The construction of Hom induces two functors on modules.

- For a fixed R-module M, the functor $\mathrm{Hom}_R(M, \bullet)$ takes a module N to $\mathrm{Hom}_R(M, N)$. The map from $N_1 \to N_2$ induces a map from $\mathrm{Hom}_R(M, N_1) \to \mathrm{Hom}_R(M, N_2)$ by composition
$$M \longrightarrow N_1 \longrightarrow N_2.$$
Thus, $\mathrm{Hom}_R(M, \bullet)$ is a functor $R\text{-}\mathbf{Mod} \to R\text{-}\mathbf{Mod}$.
- For a fixed R-module N, the functor $\mathrm{Hom}_R(\bullet, N)$ takes a module M to $\mathrm{Hom}_R(M, N)$. The map from $M_1 \to M_2$ induces a map from $\mathrm{Hom}_R(M_2, N) \to \mathrm{Hom}_R(M_1, N)$ by composition
$$M_1 \longrightarrow M_2 \longrightarrow N.$$
Thus, $\mathrm{Hom}_R(\bullet, N)$ is a functor $R\text{-}\mathbf{Mod}^{\mathrm{op}} \to R\text{-}\mathbf{Mod}$.

▶ **Bilinear Map.**

Definition 6.3.2 (Bilinear map)**.**

Let M and N be two R-modules. Then, a map $\phi : M \times N \to P$ is **bilinear** if and only if for all $m, m_1, m_2 \in M$, $n, n_1, n_2 \in N$, and $r \in R$, we have

- $\phi(m_1 + m_2, n) = \phi(m_1, n) + \phi(m_2, n)$.
- $\phi(m, n_1 + n_2) = \phi(m, n_1) + \phi(m, n_2)$.
- $\phi(rm, n) = \phi(m, rn) = r\phi(m, n)$.

For each $m \in m$, a bilinear map ϕ induces a module morphism $\phi(m, -) \in \mathrm{Hom}_R(N, P)$. Thus,
$$\{\text{bilinear map } M \times N \to P\} \;\simeq\; \mathrm{Hom}_R(M, \mathrm{Hom}_R(N, P)). \tag{6.1}$$
A bilinear map ϕ is a **perfect pairing** if the corresponding map $M \to \mathrm{Hom}_R(N, P)$ is an isomorphism.

§6.3.2 Tensor Product of Modules

In this subsection, we give a self-contained review of tensor products of arbitrary modules, with a focus on modules over $\mathbb{Z}$. For a more detailed treatment, see [AM69] or [Gat14].

Roughly speaking, tensor product is a way to study the structure of bilinear maps by given a *universal module* that represent those maps. How exactly will be apparent in the following definition.

Definition 6.3.3 (Tensor product of modules)**.**

Given two R-module M and N, the **tensor product** $M \otimes_R N$ is a module, together with a bilinear map $i : M \times N \to M \otimes_R N$ satisfying the following universal property:

"for any bilinear map $f : M \times N \to P$, there exists unique map $g : M \otimes_R N \to P$ such that $f = i \circ g$."

$$\begin{tikzcd}
 & M \otimes_R N \arrow[dr, dashed, "\exists! g"] & \\
M \times N \arrow[ur, "i"] \arrow[rr, "f"] & & P
\end{tikzcd}$$

It's not obvious whether a module with such property exists. Thus, we outline the proof.

Proposition 6.3.4 (Tensor product exists and is unique)**.**

For any R-modules M and N, the tensor product exists and unique up to isomorphism.

▶ *Proof.* The proof is in two parts.

▶ **Uniqueness.** Suppose that T_1 and T_2 satisfies the universal property of tensor product with bilinear maps $i_1 : M \times N \to T_1$ and $i_2 : M \times N \to T_2$. By using universal property on T_1, we get a map $f : T_1 \to T_2$ such that $f \circ i_1 = i_2$. Similarly, we have a map $g : T_2 \to T_1$ such that $g \circ i_2 = i_1$.

$$\begin{tikzcd} & T_1 \arrow[dd, bend right, "f"'] \\ M \times N \arrow[ur, "i_1"] \arrow[dr, "i_2"'] & \\ & T_2 \arrow[uu, bend right, "g"'] \end{tikzcd}$$

Now, by universal property, there exists unique map $t : T_1 \to T_1$ such that $t \circ i_1 = i_1$. However, we can name two such maps: id and $f \circ g$. Hence, $f \circ g = \text{id}$ and similarly, $g \circ f = \text{id}$, so (f, g) are isomorphisms.

▶ **Existence.** Given any two R-modules M and N, we need to explain how to construct the tensor product $M \otimes N$. To do that, we let

$$F = \langle m \otimes n : m \in M, n \in N \rangle$$

be the free module generated by formal symbols $m \otimes n$ for all $m \in M$ and $n \in N$.

We then now **mod out every possible relation that we want to force**. To do that, G be a free module generated by

- $(m_1 + m_2) \otimes n - (m_1 \otimes n) - (m_2 \otimes n)$,
- $m \otimes (n_1 + n_2) - (m \otimes n_1) - (m \otimes n_2)$,
- $rm \otimes n - r(m \otimes n)$, and
- $m \otimes rn - r(m \otimes n)$

for all $m, m_1, m_2 \in M$, $n, n_1, n_2 \in N$, and $r \in R$.

We left it to the reader to verify that $M \otimes_R N = F/G$, along with the map i by $i(m, n) = m \otimes n$ satisfies the universal property of tensor product. □

Given a tensor product with map $i : M \times N \to M \otimes_R N$, we denote $i(m, n)$ by $m \otimes n$. An element $m \otimes n$ is called a **pure tensor**. Since the map i is bilinear, it follows that $(m_1 + m_2) \otimes n = (m_1 \otimes n) + (m_2 \otimes n)$, $m \otimes (n_1 + n_2) = (m \otimes n_1) + (m \otimes n_2)$, and $rm \otimes n = m \otimes rn = r(m \otimes n)$.

Example 6.3.5.

We claim that $\mathbb{Z}/3 \otimes_{\mathbb{Z}} \mathbb{Z}/5 = 0$. To see why, we note that for any $m \in \mathbb{Z}/3$ and $n \in \mathbb{Z}/5$,

$$\begin{aligned} 3(m \otimes n) &= 3m \otimes n = 0 \otimes n = 0 \\ 5(m \otimes n) &= m \otimes 5n = m \otimes 0 = 0, \end{aligned}$$

so we conclude that $m \otimes n = 0$ for all $m \in \mathbb{Z}/3$ and $n \in \mathbb{Z}/5$, so the tensor product $\mathbb{Z}/3 \otimes_{\mathbb{Z}} \mathbb{Z}/5$ collapse to 0.

In some sense, tensor product allows the two modules to *talk to each other*. In this case, one module is a 3-torsion, constraining the tensor product to have 3-torsion. Similarly, another constraint the tensor product to have 5-torsion. A module with both 3-torsion and 5-torsion collapse to 0.

We now start proving basic properties of tensor product. It is tempting to use explicit construction in the proof of Proposition 6.3.4 to prove properties of tensor product. However, experience shows that for most of the time, the universal property as given in Definition 6.3.3 gives the cleanest proof.

Proposition 6.3.6 (Properties of tensor product)**.**

For any R-modules M, N, P, we have

(a) $M \otimes_R N \simeq N \otimes_R M$.

(b) $M \otimes_R (N \oplus P) \simeq (M \otimes_R N) \oplus (M \otimes_R P)$.

(c) $(M \otimes_R N) \otimes_R P = M \otimes_R (N \otimes_R P)$.

▶ *Proof.* (a) Any map from $M \otimes_R N \to T$ bijectively corresponds to a blinear map $M \times N \to T$, which in turn corresponds to a bilinear map $N \times M \to T$. This shows that $M \otimes_R N$ satisfies the universal property of $N \otimes_R M$.

(b) There is a correspondence between

- maps from $M \otimes_R (N \oplus P) \to T$.
- bilinear maps $M \times (N \oplus P) \to T$.
- maps $N \oplus P \to \mathrm{Hom}_R(M, T)$.
- pair of maps $N \to \mathrm{Hom}_R(M, T)$ and $P \to \mathrm{Hom}_R(M, T)$.
- pair of bilinear maps $M \times N \to T$ and $M \times P \to T$.
- pair of maps $M \otimes_R N \to T$ and $M \otimes_R P \to T$.
- maps $(M \otimes_R N) \oplus (M \otimes_R P) \to T$.

(c) We leave this part as an exercise. (Hint: show that a map $(M \otimes_R N) \otimes_R P \to T$ corresponds to a trilinear map $M \times N \times P \to T$.) □

Proposition 6.3.7.

For any $m, n \in \mathbb{Z}$, we have

$$\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \simeq \mathbb{Z}/\gcd(m, n)\mathbb{Z}$$

▶ *Proof.* A bilinear map $\phi : \mathbb{Z}/m \times \mathbb{Z}/n \to T$ is uniquely determined by where $(1, 1)$ is sent to. Suppose that $\phi(1, 1) = t$. Then ϕ is bilinear if and only if $mt = 0$ and $nt = 0$, or equivalently $\gcd(m, n)\, t = 0$. Therefore, picking a bilinear map $\phi : \mathbb{Z}/m \times \mathbb{Z}/n \to T$ is the same as picking a map $\mathbb{Z}/\gcd(m, n) \to T$, proving that $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \simeq \mathbb{Z}/\gcd(m, n)\mathbb{Z}$. □

Example 6.3.8.

We can compute

$$(\mathbb{Z}/6\mathbb{Z} \oplus \mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/7\mathbb{Z})$$

by expanding four terms and computing the direct sum of these. We get

$$\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \oplus 0 \oplus \mathbb{Z}/7\mathbb{Z}.$$

Since every $\mathbb{Z}$-module is a product of modules of the form $\mathbb{Z}$ or $\mathbb{Z}/n$, it follows that we can compute tensor products for any $\mathbb{Z}$-module.

§6.3.3 Exactness of tensor product.

For a fixed R-module M, tensoring with M is a functor: given a map $f : A \to B$, the map

$$\begin{aligned} A \times M &\to B \otimes_R M \\ (a, m) &\mapsto f(a) \otimes M \end{aligned}$$

is bilinear, inducing the map $f \otimes_R \mathrm{id} : A \otimes_R M \to B \otimes_R M$ which sends $a \otimes m$ to $f(a) \otimes m$.

Given a short exact sequence of R-modules

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

and another R-module M, we can form a sequence

$$0 \longrightarrow A \otimes_R M \longrightarrow B \otimes_R M \longrightarrow C \otimes_R M \longrightarrow 0,$$

but **this is not always exact**.

Example 6.3.9 (Tensor is not exact).

Suppose $R = \mathbb{Z}$, and consider

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0.$$

Tensoring by $\mathbb{Z}/2\mathbb{Z}$ gives

$$0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \xrightarrow{2} \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0.$$

The only thing wrong is the that the first map is not injective.

Still, we have a partial exactness.

Theorem 6.3.10 (Tensor is right exact).

Tensoring a short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

with an R-module M gives a right-exact sequence

$$A \otimes_R M \longrightarrow B \otimes_R M \longrightarrow C \otimes_R M \longrightarrow 0.$$

Moreover, if M is free (more generally, flat), then the adding $0 \longrightarrow$ in the front is still exact.

To prove this, we use the following lemma, whose proof is omitted. (It is not very hard.)

Lemma 6.3.11.

A sequence of R-modules

$$A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is **exact** if and only if for every R-module N,

$$0 \longrightarrow \mathrm{Hom}_R(C, N) \longrightarrow \mathrm{Hom}_R(B, N) \longrightarrow \mathrm{Hom}_R(A, N)$$

is exact.

▶ *Proof of Theorem 6.3.10.* First, we note that by universal property of tensor product, there is a bijection between $\mathrm{Hom}_R(A \otimes_R M, N)$ and bilinear map from $A \times M \to N$. Hence, (6.1), we deduce

that
$$\mathrm{Hom}_R(A \otimes_R M, N) \simeq \mathrm{Hom}_R(A, \mathrm{Hom}_R(M, N)).$$
This relation is called **tensor-hom adjunction**[1].

By the lemma, it suffices to show that for any N, the sequence
$$\mathrm{Hom}_R(C \otimes_R M, N) \longrightarrow \mathrm{Hom}_R(B \otimes_R M, N) \longrightarrow \mathrm{Hom}_R(A \otimes_R M, N) \longrightarrow 0$$
is exact. However, by the tensor-hom adjunction, this is equivalent to
$$\mathrm{Hom}_R(C, \mathrm{Hom}_R(M, N)) \longrightarrow \mathrm{Hom}_R(B, \mathrm{Hom}_R(M, N)) \longrightarrow \mathrm{Hom}_R(A, \mathrm{Hom}_R(M, N)) \longrightarrow 0$$
being exact. But this is obviously true by another direction of the above lemma. □

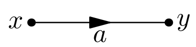
§6.3.4 Tensor Product of Chain Complexes

We now explain the how to multiply chain complexes. This is best motivated by the second major application: computing the homology of product. In particular, the tensor product of chain complexes will be rigged so that the product of $C_*^{\mathrm{cell}}(X, R)$ and $C_*^{\mathrm{cell}}(Y, R)$ is exactly $C_*^{\mathrm{cell}}(X \times Y) = R$.

To do that, we need to give the cell structure of the product of two cells.

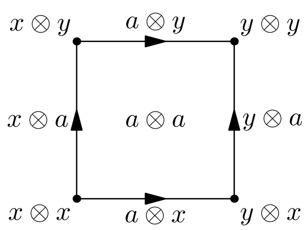
Example 6.3.12 (Cell-structure of the product).

Consider the interval $I = [0, 1]$ with CW structure



Thus, we get a cellular chain complex
$$\ldots \longrightarrow 0 \longrightarrow \mathbb{Z}\{a\} \longrightarrow \mathbb{Z}\{x, y\} \longrightarrow 0$$
with $\partial(a) = y - x$.

On the other hand, one can give a cellular structure of $I \times I$ by



In particular, one can see that the product of cell of dimension p and dimension q gives a structure of dimension $p + q$. This leads to the following definition.

Definition 6.3.13 (Tensor product of chain complexes).

Let C_* and D_* be two chain complexes of **free** R-modules. Then, the **tensor product** $C_* \otimes_R D_*$ is another chain complex given by
$$(C_* \otimes_R D_*)_n = \bigoplus_{p+q=n} C_p \otimes_R D_q$$
and boundary map
$$\partial(c_p \otimes d_q) = (\partial c_p) \otimes d_q + (-1)^p (c_p \otimes (\partial d_q)).$$

[1] If you know category theory, this is saying that $- \otimes_R M$ and $\mathrm{Hom}(-, N)$ are **adjoint functor**. We will not use this terminology in the book.

The sign $(-1)^p$ is so that the composition of two boundary maps is indeed zero. We check this by computing it directly:

$$\begin{aligned}\partial^2(c_p \otimes d_q) &= \partial\Big((\partial c_p) \otimes d_q + (-1)^p (c_p \otimes (\partial d_q))\Big) \\ &= (\partial^2 c_p) \otimes d_q + (-1)^{p-1}(\partial c_p) \otimes (\partial d_q) \\ &\qquad + (-1)^p (\partial c_p) \otimes (\partial d_q) + c_p \otimes (\partial^2 d_q) \\ &= 0.\end{aligned}$$

This will not be the last time that we will see sign issues.

Warning 6.3.14.

This construction only works for complexes of **free** modules (which is the case for $C_*^{\text{cell}}(X, R)$ and $S_*(X, R)$). In the next section, we will explain why this definition is bad for non-free modules and how to fix it.

In the general case, we have that

Proposition 6.3.15.

Given two CW complexes X and Y, then we have

$$C_*^{\text{cell}}(X \times Y, R) \simeq C_*^{\text{cell}}(X, R) \otimes_R C_*^{\text{cell}}(Y, R).$$

We could prove this proposition now. However, we will show how compute the homology of products for **any topological space**, not just CW Complex. We will state and prove more general theorem at Theorem 6.6.1.

Once we have the chain complex, the next question is how we can extract the homology. Unfortunately, there is no recipe to compute $H_*(C_* \otimes_R D_*)$ for any general chain complexes C_* and D_*. However, if C_* and D_* are chain complexes of **free** R-modules (which is the case here), we can compute $H_*(C_* \otimes_R D_*)$ in terms of $H_*(C_*)$ and $H_*(D_*)$.

§6.4 Homological Algebra

§6.4.1 Free Resolution

We now explain how, given two chain complexes $C_*(X, R)$ and $D_*(Y, R)$ of free R-modules, one can compute $H_*(C_* \otimes_R D_*)$ from $H_*(C_*)$ and $H_*(D_*)$.

Definition 6.4.1 (Quasi-isomorphism)**.**

A map $f_* : C_* \to D_*$ of chain complexes

$$\begin{tikzcd} \dots \arrow[r] & C_2 \arrow[r] \arrow[d, "f_2"] & C_1 \arrow[r] \arrow[d, "f_1"] & C_0 \arrow[r] \arrow[d, "f_0"] & \dots \\ \dots \arrow[r] & D_2 \arrow[r] & D_1 \arrow[r] & D_0 \arrow[r] & \dots \end{tikzcd}$$

is called a **quasi-isomorphism** if it induces an isomorphism on all homology groups.

Example 6.4.2.

If $f : X \to Y$ is a homotopy equivalence, then $C_*(f) : C_*(X, R) \to C_*(Y, R)$ is a quasi-isomorphism.

Example 6.4.3.

Let $C_* = \mathbb{Z} \xrightarrow{\cdot 5} \mathbb{Z}$ and $D = \mathbb{Z}/5\mathbb{Z}$. The map $f : C_* \to D_*$ given by

$$\begin{tikzcd}
\dots \arrow[r] & 0 \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r, "\cdot 5"] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & 0 \arrow[r] & \dots \\
\dots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \mathbb{Z}/5\mathbb{Z} \arrow[r] & 0 \arrow[r] & \dots
\end{tikzcd}$$

is a quasi-isomorphism.

Tensoring the recent example with D_* gives the map $f \otimes \mathrm{id} : C_* \otimes_{\mathbb{Z}} D_* \to D_* \otimes_{\mathbb{Z}} D_*$ given by

$$\begin{tikzcd}
\dots \arrow[r] & 0 \arrow[r] \arrow[d] & \mathbb{Z}/5\mathbb{Z} \arrow[r, "\cdot 5"] \arrow[d] & \mathbb{Z}/5\mathbb{Z} \arrow[r] \arrow[d] & 0 \arrow[r] & \dots \\
\dots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \mathbb{Z}/5\mathbb{Z} \arrow[r] & 0 \arrow[r] & \dots,
\end{tikzcd}$$

which is **not** a quasi-isomorphism. This is an explicit example of why tensoring with chain complex not consisting of free modules behave poorly.

Definition 6.4.4.

For any R-module M, a **free resolution** of M is a chain complex of free modules with quasi-isomorphism to a simple chain complex with just M, as shown in diagram below:

$$\begin{tikzcd}
\dots \arrow[r] & F_2 \arrow[r] \arrow[d] & F_1 \arrow[r] \arrow[d] & F_0 \arrow[r] \arrow[d, "f"] & 0 \arrow[r] \arrow[d] & \dots \\
\dots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & M \arrow[r] & 0 \arrow[r] & \dots
\end{tikzcd},$$

where F_i are free modules.

Any module admits at least one free resolution, but the resolution is not unique. We saw one free resolution of $\mathbb{Z}/5\mathbb{Z}$ in Example 6.4.3. Another one is

$$\begin{array}{ccccccc}
\dots \longrightarrow \mathbb{Z} & \longrightarrow & \mathbb{Z} \oplus \mathbb{Z} & \longrightarrow & \mathbb{Z} \longrightarrow \dots \\
1 & \mapsto & (1,0) & & \\
 & & (1,0) & \mapsto & 0 \\
 & & (0,1) & \mapsto & 5
\end{array}.$$

If R is a field (e.g. $\mathbb{Q}$ or $\mathbb{F}_p$), then every module is free, hence its own free resolution.

If R is a PID, then every finitely-generated module admits a two-term free resolution. This is because for any module M, one can create a free resolution from any surjective map $f : R^{\oplus n} \to M$, and we can make a free resolution

$$\begin{tikzcd}
\dots \arrow[r] & 0 \arrow[r] \arrow[d] & \operatorname{Ker} f \arrow[r] \arrow[d] & R^{\oplus n} \arrow[r] \arrow[d, "f"] & 0 \arrow[r] \arrow[d] & \dots \\
\dots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & M \arrow[r] & 0 \arrow[r] & \dots
\end{tikzcd},$$

This works because by PID structure theorem, any submodule of free module is free.

In general, every module M has a free resolution. Indeed, we can first take a free module M_0 with a surjection $\varphi : M_0 \to M$. Then take a free module M_1 with a surjection $\partial_1 : M_1 \to \operatorname{Ker} \varphi$. Take a free module M_2 with a surjection $\partial_2 : M_2 \to \operatorname{Ker} \partial_1$, and so on. The resulting complex

$$\cdots \xrightarrow{\partial_3} M_2 \xrightarrow{\partial_2} M_1 \xrightarrow{\partial_1} M_0 \longrightarrow 0 \longrightarrow \cdots$$

is a free resolution of M by construction. (The process may not terminate, but that's fine because free resolutions need not be finite.)

In this class, we will be mostly interested in the case where R is a field or a PID.

We now state the main property of free resolutions.

Theorem 6.4.5 (Fundamental Theorem of Homological Algebra).

Suppose that $f : M \to N$ is a map of R-modules. Suppose we are given free resolutions

$$\begin{tikzcd} \dots \arrow[r] & F_2 \arrow[r] \arrow[d] & F_1 \arrow[r] \arrow[d] & F_0 \arrow[d, "f"] \\ \dots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & M \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \dots \arrow[r] & E_2 \arrow[r] \arrow[d] & E_1 \arrow[r] \arrow[d] & E_0 \arrow[d, "g"] \\ \dots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & N \end{tikzcd}.$$

Then, there exists a map $f_* : F_* \to E_*$ of chain complexes of R-modules such that $H_0(f_*) : M \to N$ is the same as f.

Furthermore, f_* is unique up to chain homotopy.

The proof of this theorem is straightforward but tedious. You are encouraged to skim through this quickly and convince yourself that there is nothing hard.

▶ *Proof.* We use the following properties labeled (♠) of free modules: if F is a free module, then for any surjective map $f : P \to Q$, it is possible to lift any map $\pi : F \to Q$ to $\widetilde{\pi} : F \to P$

$$\begin{tikzcd} & F \arrow[dl, dashed, "\exists\widetilde{\pi}"'] \arrow[d, "\psi"] \\ P \arrow[r, two heads, "f"'] & Q \end{tikzcd}.$$

(Aside: Modules with this property are called **projective modules**, and this is the only condition of F_i that we need to make this theorem works, so instead of free resolutions, you can use a more general projective resolutions.) To verify this property, simply send a generator $e_i \in F$ to any element in $f^{-1}(\{\psi(e_i)\})$.

Now, we prove the theorem.

Existence of a lift. First, we show existence by construction the map $\widetilde{\phi}_i : F_i \to E_i$ by induction. In the base case, consider the following diagram

$$\begin{tikzcd} F_0 \arrow[r, two heads] \arrow[d, dashed] \arrow[dr] & M \arrow[d, "\phi"] \\ E_0 \arrow[r, two heads] & N \end{tikzcd}$$

Thus, we use (♠) to find the map $\widetilde{\phi}_0$ such that the diagram above commute.

Now, for the inductive step, suppose we have $\widetilde{\phi}_{i-1} : F_{i-1} \to E_{i-1}$. Let $K_{i-1} = \operatorname{Ker}(F_{i-1} \to F_{i-2})$ and $L_{i-1} = \operatorname{Ker}(E_{i-1} \to E_{i-2})$ Then, $\widetilde{\phi}_{i-1}$ induces the map $\widetilde{\phi}_{i-1} : K_{i-1} \to L_{i-1}$. Then, we have the diagram

$$\begin{tikzcd} F_i \arrow[r, two heads] \arrow[d, dashed] \arrow[dr] & L_{i-1} \arrow[d, "\widetilde{\phi}_{i-1}"] \\ E_i \arrow[r, two heads] & L_{i-1} \end{tikzcd},$$

and again, we can use (♠) to find $\widetilde{\phi}_i : F_i \to E_i$ so that the above diagram commutes.

Any two lifts are chain homotopic. We have to construct the chain homotopy. Consider two lifts $\psi, \pi : F_* \to F'_*$. Let $\ell = \psi - \pi$. We need h such that $\partial h - h\partial = \ell$. We, again, do this by induction. Ths base case is to find h_0 so that the following diagram commutes:

$$\begin{tikzcd} & & F_0 \arrow[r, two heads] \arrow[dll, dashed, "h_0"'] \arrow[dl, dashed] \arrow[d, "\ell_0"] & M \arrow[d, "0"] \\ E_1 \arrow[r, two heads] & L_1 \arrow[r, hook] & E_0 \arrow[r, two heads] & N \end{tikzcd}$$

This is done in two steps. First, note that the composition $M \to E_0 \to N$ is zero map, so that composition factors through L_1. Therefore, we get $F_0 \to L_1$. Then, use (♠) again to find $F_0 \to E_1$. Since the term $h\partial$ vanishes, we get that the chain homotopy condition is verified at 0.

Now, the inductive step is similar. We have the diagram

$$\begin{tikzcd} & & F_i \arrow[dll, dashed, "h_i"'] \arrow[dl, dashed] \arrow[d, "\ell_i + h_{i-1}\partial_i"] \\ E_{i+1} \arrow[r, two heads] & L_{i+1} \arrow[r, hook] & E_i \end{tikzcd}$$

We have to construct h_i such that $\partial_{i+1}h_i - h_{i-1}\partial_i = \ell_i$. To do this, we first note that

$$\begin{aligned} \partial_i\Big(\ell_i + h_{i-1}\partial_i\Big) &= \partial_i\ell_i + \partial_i h_{i-1}\partial_i \\ &= \ell_{i-1}\partial_i + \partial_i h_{i-1}\partial_i && (\ell \text{ commutes with boundary maps}) \\ &= (\ell_{i-1} + \partial_i h_{i-1})\partial_i \\ &= h_{i-2}\partial_{i-1}\partial_i && (\text{chain homotopy definition}) \\ &= 0, \end{aligned}$$

so $\ell_i + h_{i-1}\partial_i$ factors through L_{i+1}, getting the map $F_i \to L_{i+1}$. Finally, use (♠) to lift it to h_i, giving the desired chain homotopy condition. □

§6.4.2 Derived Category

The above theorem leads us to the world of **homological algebra**. This is a study of the **derived category** of R-**Mod**, $D(R\text{-}\mathbf{Mod})$.

Definition 6.4.6.

If R is a commutative ring, then $D(R\text{-}\mathbf{Mod})$ (sometimes denoted $D(R)$) is a category such that

- the objects are chain complex of R-modules.
- the morphisms include all morphisms of chain complex of R-modules, but it also includes formal inverses to quasi-isomorphisms.

This is a technical mess to set up (we will run into set-theoretic issues). It is similar to when you form $\mathbb{Z}$ form $\mathbb{N}$ by adding additive inverses. We will spare the details of the construction, and use (without proof) the existence of it.

There is a tensor product and an internal hom in $D(R)$. The tensor product of C_* and D_* in $D(R)$ is denoted $C_* \otimes_R^{\mathbb{L}} D_*$. We will not tell you how to define it, but one can compute it as follows.

Definition 6.4.7.

Given $C_*, D_* \in D(R\text{-}\mathbf{Mod})$, we define the **derived tensor product** $C_* \otimes_R^{\mathbb{L}} D_*$ as follows.

If either C_* or D_* is a chain complex of free R-modules, then

$$C_* \otimes_R^{\mathbb{L}} D_* = C_* \otimes_R D_*.$$

To calculate $\otimes^{\mathbb{L}}$ is general, we can replace D_* by a quasi-isomorphic chain complex of free module.

We blackbox the highly nontrival fact that this is well-defined (in particular, it does not depend on a choice of free resolution of D_*). See [Wei94] for more details about derived category.

§6.4.3 The Tor Functor

The Tor functor is the homology group of the derived tensor product. Using derived category, we can define it as follows: if M and N are R-modules, viewed as degree 0 chain complexes, then we define the **Tor functor**

$$\mathrm{Tor}_i(M,N) = \mathrm{Tor}_i^R(M,N) := H_i(M \otimes_R^{\mathbb{L}} N).$$

Alternatively, we can define it without reference to derived category.

Definition 6.4.8.

Given two R-modules M and N, we define $\mathrm{Tor}_i^R(M,N)$ as follows: if F_* is a free resolution of N, then

$$\mathrm{Tor}_i^R(M,N) = H_i(M \otimes_R F_*)$$

It's not obvious that this does not depend on the choice of F_*. However, we can quickly show this by using the fundamental theorem of homological algebra Theorem 6.4.5.

▶ *Proof that* Tor *does not depend on the choice of free resolution.* Consider two free resolutions F_* and F'_* for N. Thus, by the fundamental theorem of homological algebra (Theorem 6.4.5), the identity map $\mathrm{id} : N \to N$ lifts to $\phi : F_* \to F'_*$ and $\psi : F'_* \to F_*$. Then, $\phi \circ \psi : F_* \to F_*$ is a lift of id as well. However, the identity map is one such liftlift of id, so $\phi \circ \psi$ is chain homotopic to identity.

Tensoring with M, we get two maps $\phi_* : M \otimes F_* \to M \otimes F'_*$ and $\psi_* : M \otimes F'_* \to M \otimes F_*$. The chain homotopic condition is preserved through tensoring with M. Thus, $\phi_* \circ \psi_*$ is chain homotopic to identity. Similarly, $\psi_* \circ \phi_*$ is chain homotopic to identity. Thus, by using Proposition 4.2.5, $H_n(\phi_*)$ and $H_n(\psi_*)$ provide isomorphism between $H_n(M \otimes F_*) \simeq H_n(M \otimes F'_*)$. This implies that $\mathrm{Tor}_n^R(M,N)$ gives the same result for two resolutions F_* and F'_*. □

Example 6.4.9.

We compute $\mathbb{Z}/4\mathbb{Z}$:

$$\begin{aligned}(\mathbb{Z}/2\mathbb{Z}) \otimes_{\mathbb{Z}}^{\mathbb{L}} (\mathbb{Z}/4\mathbb{Z}) &\simeq \mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}}^{\mathbb{L}} (\cdots \to \mathbb{Z} \xrightarrow{4} \mathbb{Z} \to 0)\\ &\simeq (\cdots \to 0 \to \mathbb{Z}/2\mathbb{Z} \xrightarrow{4} \mathbb{Z}/2\mathbb{Z} \to 0)\\ &\simeq (\cdots \to 0 \to \mathbb{Z}/2\mathbb{Z} \xrightarrow{0} \mathbb{Z}/2\mathbb{Z} \to 0),\end{aligned}$$

which will get the same answer as the previous way.

The above example shows that it's not hard to compute Tor in simple cases like $\mathbb{Z}$-modules, and in fact one can just memorize it. We can show that Using the recipe in finding the free resolution, we then find that

Proposition 6.4.10.

(a) If M and N are free R-modules, then

$$\mathrm{Tor}_i^R(M,N) = \begin{cases} M \otimes_R N & \text{if } i = 0 \\ 0 & \text{if } i \neq 0 \end{cases}$$

(b) If $R = \mathbb{Z}$, then

$$\operatorname{Tor}_i^{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) = \begin{cases} \mathbb{Z}/\gcd(m,n)\mathbb{Z} & \text{if } i = 0, 1 \\ 0 & \text{if } i \neq 0, 1 \end{cases}$$

$$\operatorname{Tor}_i^{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) = \begin{cases} \mathbb{Z}/n\mathbb{Z} & \text{if } i = 0 \\ 0 & \text{if } i \neq 0. \end{cases}$$

Here are some properties of Tor.

Proposition 6.4.11 (Properties of Tor)**.**

We have

(a) $\operatorname{Tor}_i^R(M, N) = \operatorname{Tor}_i^R(N, M)$. In particular, one can compute $\operatorname{Tor}_i^R(M, N)$ by taking a free resolution of M as well.

(b) $\operatorname{Tor}_i^R(M \oplus M', N) \simeq \operatorname{Tor}_i^R(M, N) \oplus \operatorname{Tor}_i^R(M', N)$.

(c) Given a short exact sequence of R-modules $0 \to A \to B \to C \to 0$, we have the long exact sequence

$$\begin{tikzcd} \dots \arrow[r] & \operatorname{Tor}_2^R(A, M) \arrow[r] & \operatorname{Tor}_2^R(B, M) \arrow[r] & \operatorname{Tor}_2^R(C, M) \arrow[dll] \\ & \operatorname{Tor}_1^R(A, M) \arrow[r] & \operatorname{Tor}_1^R(B, M) \arrow[r] & \operatorname{Tor}_1^R(C, M) \arrow[dll] \\ & A \otimes_R M \arrow[r] & B \otimes_R M \arrow[r] & C \otimes_R M \arrow[r] & \dots \end{tikzcd}$$

▶ *Proof.* (a) Intuitively, this should be true because tensor product is symmetric ($M \otimes_R N = N \otimes_R M$), but this turns out to be a very difficult one to prove. Roughly speaking, we take a free resolution F_* and G_* of both M and N. Then, take their product $F_i \otimes_R G_j$ to get a double complex (that looks like an infinite grid). Then, one manipulates homology groups of the double complex and proves the result (typically, this is done via spectral sequence). We refer the reader to [Wei94, Thm. 2.7.2].

(b) This follows from that if F_* and F'_* are free resolutions of M and M', then $F_* \oplus F'_*$ is a free resolution of $M \oplus M'$.

(c) The short exact sequence $0 \to A \to B \to C \to 0$ induces a short exact sequence of chain complex

$$0 \longrightarrow A \otimes_R F_* \longrightarrow B \otimes_R F_* \longrightarrow C \otimes_R F_* \longrightarrow 0,$$

which is exact because each module in F_* is free. The result now follows from the homology long exact sequence (Theorem 4.3.11). □

§6.5 Universal Coefficient Theorem

§6.5.1 Statement and Examples

With the setup of homological algebra and Tor functor, we are ready to state a big theorem.

Theorem 6.5.1 (Universal Coefficient Theorem).

Let R be a PID (including fields). Let C_* and D_* be a chain complex of R-modules such that C_* is a complex of **free** R-modules. Then, there is an isomorphism of R-modules

$$H_n(C_* \otimes_R D_*) \simeq \left(\bigoplus_{p+q=n} H_p(C_*) \otimes_R H_q(D_*) \right) \oplus \left(\bigoplus_{p+q=n-1} \mathrm{Tor}_1^R(H_p(C_*), H_q(D_*)) \right).$$

This theorem have two powerful consequences.

First, it provides a formula to compute $H_n(X, R)$ from $H_n(X, \mathbb{Z})$. To explain this, we note that since the singular chain complex $S_n(X, \mathbb{Z})$ is a free $\mathbb{Z}$-module, we have

$$S_n(X, R) = S_n(X, \mathbb{Z}) \otimes_{\mathbb{Z}} R,$$

and one can check that the boundary map on $S_n(X, R) \to S_{n-1}(X, R)$ is the same as tensoring the boundary map $S_n(X, \mathbb{Z}) \to S_{n-1}(X, \mathbb{Z})$ with R. Thus, as chain complexes, we have that

$$S_*(X, R) = S_n(X, \mathbb{Z}) \otimes_{\mathbb{Z}}^{\mathbb{L}} R,$$

where R is regarded as the chain complex $\ldots 0 \to 0 \to R \to 0$. Hence, by plugging in $R = \mathbb{Z}$ and $D_* = R$, we get the following corollary:

Corollary 6.5.2.

Let X be a topological space. Then,

$$H_n(X, R) \simeq (H_n(X) \otimes_{\mathbb{Z}} R) \oplus \mathrm{Tor}_1^{\mathbb{Z}}(H_{n-1}(X), R).$$

(Note: this is true for any ring R.)

In particular, **one can determine $H_n(X, R)$ from just $H_n(X, \mathbb{Z})$**.

Warning 6.5.3 (This is not natural.).

Neither isomorphisms in Theorem 6.5.1 or Corollary 6.5.2 are natural. We explain why universal coefficient theorem cannot be made natural.

Consider two maps $f : \mathbb{RP}^2 \to S^2$ and $g : \mathbb{RP}^2 \to S^2$ where f is defined by collapsing $\mathbb{RP}^1$ to a point, while g is defined by sending everything to one point. Then we have the induced maps on $\mathbb{Z}$-homology

$$H_1(f), H_1(g) : \mathbb{F}_2 \to 0$$
$$H_2(f), H_2(g) : 0 \to \mathbb{Z},$$

which means that $H_*(f)$ and $H_*(g)$ are the same.

However, if we consider $\mathbb{F}_2$-homology, then $H_2(g, \mathbb{F}_2) : \mathbb{F}_2 \to \mathbb{F}_2$ is a zero map because at the chain level it is a zero map. On the other hand, $H_2(f, \mathbb{F}_2)$ is identity. In particular, **even though f and g gives the same map on $\mathbb{Z}$-homology, it does not give the same map on $\mathbb{F}_2$-homology**. This tells us that the isomorphism in Corollary 6.5.2 cannot be made natural.

The second application of this theorem is to compute homology groups $H_*(X \times Y, R)$ from $H_*(X, R)$ and $H_*(Y, R)$. In the case of chain complexes, we saw that by Proposition 6.3.15, the cellular chain complex $C_*^{\text{cell}}(X \times Y, R)$ is the same as $C_*^{\text{cell}}(X, R) \otimes_R C_*^{\text{cell}}(Y, R)$, and we can use Theorem 6.5.1 to compute homology of tensor product of chain complex. It turns out that this method works for a general topological spaces, and we will see this in Theorem 6.6.1.

Before proving this theorem, we demonstrate it with some examples. Recalling that

$$H_n(\mathbb{RP}^2) = \begin{cases} \mathbb{Z} & n = 0 \\ \mathbb{Z}/2\mathbb{Z} & n = 1 \\ 0 & n \geq 2 \end{cases},$$

we compute

$$\begin{aligned} H_2(\mathbb{RP}^2, \mathbb{F}_2) &\simeq (H_2(\mathbb{RP}^2) \otimes_{\mathbb{Z}} \mathbb{F}_2) \oplus \mathrm{Tor}_1^{\mathbb{Z}}(H_1(\mathbb{RP}^2), \mathbb{F}_2) \\ &\simeq 0 \oplus \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \\ &\simeq 0 \oplus \mathbb{Z}/2\mathbb{Z} \\ &= \mathbb{Z}/2\mathbb{Z}. \end{aligned}$$

On the other hand, the first term may be the one that contributes. For example,

$$\begin{aligned} H_2(S^2, \mathbb{F}_2) &\simeq (H_2(S^2) \otimes_{\mathbb{Z}} \mathbb{F}_2) \oplus \mathrm{Tor}_1^{\mathbb{Z}}(H_1(S^2), \mathbb{F}_2) \\ &\simeq (\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{F}_2) \oplus \mathrm{Tor}_1^{\mathbb{Z}}(0, \mathbb{Z}/2\mathbb{Z}) \\ &= \mathbb{Z}/2\mathbb{Z} \oplus 0 \\ &= \mathbb{Z}/2\mathbb{Z}. \end{aligned}$$

§6.5.2 Proof

We show two proofs of the Universal Coefficient theorem: one uses derived category, and the other does not.

▶ *Proof using Derived Category.* Note the following observations.

- Universal Coefficient Theorem is true if both C_* and D_* are concentrated in degree 0 (and hence degree a and b for $a, b \in \mathbb{Z}$). This follows from the fact that any R-module has a two-term free resolution (and uses the fact that R is a PID).
- If Universal Coefficient Theorem holds for $C_* \otimes_R^{\mathbb{L}} D_*$ and $C'_* \otimes_R^{\mathbb{L}} D_*$, then it holds for $(C_* \oplus C'_*) \otimes_R^{\mathbb{L}} D_*$.

Thus, to prove the Universal Coefficient theorem, it suffices to express C_* and D_* as a direct sum of chain complexes, each concentrated in a single degree (i.e., have one nonzero object). In other words, it suffices to show that every chain complex C_* in $D(R\text{-}\mathbf{Mod})$ is (quasi-)isomorphic to direct sum of chain complexes, each concentrated in a single degree.

To prove the claim, let $K_n = \mathrm{Ker}(C_n \to C_{n-1})$. We consider the morphism between C_* and the complex K_* with boundary map all zeros. Then we compose it with the projection $K_n \to H_n(C_*)$, getting the following diagram:

$$\begin{tikzcd}
\dots \arrow[r] & C_3 \arrow[r, "\partial"] \arrow[d, "\partial"] & C_2 \arrow[r, "\partial"] \arrow[d, "\partial"] & C_1 \arrow[r] \arrow[d, "\partial"] & \dots \\
\dots \arrow[r] & K_2 \arrow[r, "0"] \arrow[d] & K_1 \arrow[r, "0"] \arrow[d] & K_0 \arrow[r] \arrow[d] & \dots \\
\dots \arrow[r] & H_2(C_*) \arrow[r, "0"] & H_1(C_*) \arrow[r, "0"] & H_0(C_*) \arrow[r] & \dots
\end{tikzcd}$$

Clearly, the overall composition is a quasi-isomorphism because the homology groups are isomorphic. Thus, we replace the complex C_* with $H_n(C_*)$ with boundary maps all zero. The latter complex clearly can be written as the desired direct sum of $\cdots \to 0 \to H_n(C_*) \to 0 \to \ldots$. □

▶ *Proof without Derived Category.* Let $L_n = \operatorname{Ker}(D_n \to D_{n-1})$. We have the short exact sequence

$$0 \to L_* \to D_* \to D_*/L_* \to 0,$$

where both L_* and D_*/L_* are regarded as chain complexes whose boundary maps are all 0 and $L_* \to D_*$ are inclusion maps. Since C_* is free, for each n, we get an exact sequence

$$0 \to C_* \otimes_R L_* \to C_* \otimes_R D_* \to C_* \otimes_R D_*/L_* \to 0,$$

and taking long exact sequence gives

$$H_n(C_* \otimes_R L_*) \to H_n(C_* \otimes_R D_*) \to H_n(C_* \otimes_R D_*/L_*) \to H_{n-1}(C_* \otimes_R L_*). \tag{6.2}$$

Note that L_* and D_*/L_* can be expressed as a direct sum of chain complexes concentrated in a single degree. Following the previous proof, we see that UCT holds for L_* and D_*/L_*. Furthermore, there is no Tor_1 terms because C_n is free for all n. Thus, we deduce that $H_n(C_* \otimes_R L_*) = \bigoplus_i (C_i \otimes_R H_{n-i}(L_*))$ and similarly for $H_n(C \otimes_R D_*/L_*)$.

Let

$$\delta_n : \bigoplus_i (H_{n-i}(C_*) \otimes_R D_{i+1}/L_{i+1}) \to \bigoplus_i (H_{n-i}(C_*) \otimes_R L_i)$$

be the boundary map in (6.2). By splicing (6.2) into short exact sequences, we get a short exact sequenece

$$0 \to \operatorname{Coker} \delta_n \to H_n(C_* \otimes_R D_*) \to \operatorname{Ker} \delta_{n-1} \to 0. \tag{6.3}$$

Note taht $D_{i+1}/L_{i+1} = \operatorname{Im}(D_{i+1} \to D_i)$, which implies (by right-exactness of tensor product) that

$$\operatorname{Coker} \delta_n = \bigoplus_i \left(H_{n-i}(C_*) \otimes_R \frac{L_i}{\operatorname{Im}(D_{i+1} \to D_i)} \right) = \bigoplus_i (H_{n-i}(C_*) \otimes_R H_i(D_*)). \tag{6.4}$$

We now idenfity $\operatorname{Ker} \delta_{n-1}$. To do this, note that the short exact sequence $0 \to D_{i+1}/L_{i+1} \to L_i \to H_i(D_*) \to 0$ gives the long exact sequence

$$\begin{aligned}\operatorname{Tor}_1^R(H_{n-1-i}(C_*), H_i(D_*)) &\to H_{n-1-i}(C_*) \otimes_R D_{i+1}/L_{i+1} \\ &\to H_{n-1-i}(C_*) \otimes_R L_i \to H_{n-1-i}(C_*) \otimes_R H_i(D_*) \to 0,\end{aligned}$$

and taking direct sum across all $i \in \mathbb{Z}$ gives

$$\operatorname{Ker} \delta_{n-1} = \bigoplus_{p+q=n-1} \operatorname{Tor}_1^R(H_p(C_*), H_q(D_*)). \tag{6.5}$$

Thus, plugging in (6.4) and (6.5) into (6.3) gives the short exact sequence

$$0 \to \bigoplus_{p+q=n} H_p(C_*) \otimes_R H_q(D_*) \to H_n(C_* \otimes_R D_*) \to \bigoplus_{p+q=n-1} \operatorname{Tor}_1^R(H_p(C_*), H_q(D_*)) \to 0.$$

Finally, we have to show that this exact sequence splits. To do this, we use Problem 4.A ((iii) implies (i)) by finding the map $H_n(C_* \otimes_R D_*) \to \bigoplus_{p+q=n} H_p(C_*) \otimes_R H_q(D_*)$. Let $K_i = \operatorname{Ker}(C_i \to C_{i-1})$. By the first isomorphism theorem $C_i/K_i \simeq \operatorname{Im}(C_{i+1} \to C_i) \subset C_i$, which is a free module. Therefore, the exact sequence $0 \to K_i \to C_i \to C_i/K_i \to 0$ splits, defining the map $t : C_i \to K_i$. Tensoring and taking homology gives the map $\bar{t} : H_n(C_* \otimes D_*) \to H_n(K_* \otimes D_*)$, which induces the map $\bar{t} : H_n(C_* \otimes D_*) \to \operatorname{Coker} \delta_n$. It is not very hard to see that this works. □

§6.6 Acyclic Models and Eilenberg-Zilber Theorem

§6.6.1 Statement and Example

We will now prove a theorem that allows you to compute homology of product. We did this in CW complexes, but we will now make a general statement that works for any topological space.

Recall that for any topological space X, $S_*(X, R)$ is the chain complex with the n-th term being $R\{\operatorname{Sing}_n(X)\}$.

Theorem 6.6.1 (Eilenberg-Zilber Theorem)**.**

For any pair of topological spaces X and Y, we have

$$S_*(X,R)\otimes^{\mathbb{L}}_R S_*(Y,R)\simeq S_*(X\times Y,R),$$

where the isomorphism is in the derived category.

Example 6.6.2.

We compute $H_n(\mathbb{RP}^2\times\mathbb{RP}^2,\mathbb{F}_2)$. Recall that

$$H_n(\mathbb{RP}^2,\mathbb{F}_2)=\begin{cases}\mathbb{Z}/2\mathbb{Z} & n=0,1,2\\ 0 & n\geq 3.\end{cases}$$

Thus, by Eilenberg-Zilber Theorem, we have $H_n(\mathbb{RP}^2\times\mathbb{RP}^2)=H_n(S_*(\mathbb{RP}^2)\otimes^{\mathbb{L}}_{\mathbb{F}_2}S_*(\mathbb{RP}^2))$. Thus, by universal coefficient theorem,

$$\begin{aligned}H_n(\mathbb{RP}^2\times\mathbb{RP}^2,\mathbb{F}_2)&=\bigoplus_{p+q=n}H_p(\mathbb{RP}^2,\mathbb{F}_2)\otimes_{\mathbb{F}_2}H_q(\mathbb{RP}^2,\mathbb{F}_2)\\&\quad\underbrace{\oplus\bigoplus_{p+q=n-1}\operatorname{Tor}_1^{\mathbb{F}_2}(H_p(\mathbb{RP}^2,\mathbb{F}_2),H_q(\mathbb{RP}^2,\mathbb{F}_2))}_{=0}\\&=\bigoplus_{\substack{p\leq 2\\ q\leq 2\\ p+q=n}}\mathbb{F}_2.\end{aligned}$$

Hence, we have that

$$\begin{aligned}H_0(\mathbb{RP}^2\times\mathbb{RP}^2,\mathbb{F}_2)&=\mathbb{F}_2\\H_1(\mathbb{RP}^2\times\mathbb{RP}^2,\mathbb{F}_2)&=\mathbb{F}_2\oplus\mathbb{F}_2\\H_2(\mathbb{RP}^2\times\mathbb{RP}^2,\mathbb{F}_2)&=\mathbb{F}_2\oplus\mathbb{F}_2\oplus\mathbb{F}_2\\H_3(\mathbb{RP}^2\times\mathbb{RP}^2,\mathbb{F}_2)&=\mathbb{F}_2\oplus\mathbb{F}_2\\H_4(\mathbb{RP}^2\times\mathbb{RP}^2,\mathbb{F}_2)&=\mathbb{F}_2\\H_n(\mathbb{RP}^2\times\mathbb{RP}^2,\mathbb{F}_2)&=0\qquad\text{for all } n\geq 5.\end{aligned}$$

§6.6.2 Proof

Let us now explain how one proves this theorem. The key tool is Alexander-Whitney map. The **Alexander-Whitney map** AW is a quasi-isomorphism

$$S_*(X\times Y,R)\longrightarrow S_*(X,R)\otimes_R S_*(Y,R).$$

To define this map, it suffices to define degree n part, which is a map

$$\mathrm{AW}_n:R\{\mathrm{Sing}_n(X\times Y)\}\longrightarrow\bigoplus_{p+q=n}R\{\mathrm{Sing}_p(X)\}\otimes_R R\{\mathrm{Sing}_q(Y)\},$$

and so it suffices to define, for each p,q,

$$\mathrm{AW}_{(p,q)}:R\{\mathrm{Sing}_{p+q}(X\times Y)\}\longrightarrow R\{\mathrm{Sing}_p(X)\}\otimes_R R\{\mathrm{Sing}_q(Y)\},$$

which send a generator

$$(\sigma:\Delta^{p+q}\to X\times Y)=(\sigma_1:\Delta^{p+q}\to X,\ \sigma_2:\Delta^{p+q}\to Y)$$

to

$$(\sigma_1|_{\text{front } p\text{-face}}) \otimes (\sigma_2|_{\text{back } q\text{-face}}).$$

Here, the front p-face of Δ^n is the map $\Delta^p \hookrightarrow \Delta^n$ sending $(e_0, \dots, e_p)$ to $(e_0, e_1, \dots, e_p, 0, 0, \dots, 0)$. Similarly, the back q-face of Δ^n is the map $\Delta^q \hookrightarrow \Delta^n$ sending $(e_0, \dots, e_q)$ to $(0, 0, \dots, 0, e_0, e_1, \dots, e_q)$.

We have to check that this map is a chain map (in other words, AW commutes with ∂). This is a fairly uninspiring calculation, but we have to do it to make sure that everything works.

Suppose that $\sigma : \Delta^n \to X \times Y$ are $t_i = (x_i, y_i)$ for $i \in \{0, 1, \dots, n\}$. Then we compute

$$\begin{aligned}
\mathrm{AW}(\sigma) &= \sum_{p=0}^{n} [x_0, x_1, \dots, x_p] \otimes [y_p, y_{p+1}, \dots, n] \\
\partial(\mathrm{AW}(\sigma)) &= \sum_{p=0}^{n} \Bigg(\left(\sum_{i=0}^{p} (-1)^i [x_0 \dots, \widehat{x_i}, \dots x_p] \right) \otimes [y_p, y_{p+1}, \dots, y_n] \\
&\qquad + (-1)^p \cdot [x_0, \dots, x_p] \otimes \left((-1)^j \sum_{j=0}^{n-p} [y_p, \dots, \widehat{y_{p+j}}, \dots, y_n] \right) \Bigg) \\
&= \sum_{p=0}^{n} \Bigg(\left(\sum_{i=0}^{p} (-1)^i [x_0 \dots, \widehat{x_i}, \dots x_p] \right) \otimes [y_p, y_{p+1}, \dots, y_n] \\
&\qquad + [x_0, \dots, x_p] \otimes \left(\sum_{j=p}^{n} (-1)^j [y_p, \dots, \widehat{y_j}, \dots, y_n] \right) \Bigg) \\
&= \sum_{i=0}^{n} (-1)^i \sum_{p=i}^{n} [x_0, \dots, \widehat{x_i}, \dots, x_p] \otimes [y_p, \dots, y_n] \\
&\qquad + \sum_{j=0}^{n} (-1)^j \sum_{p=0}^{j} [x_0, \dots, x_p] \otimes [y_p, \dots, \widehat{y_j}, \dots, y_n] \\
&= \sum_{i=0}^{n} (-1)^i \Bigg(\sum_{p=0}^{i} [x_0, \dots, x_p] \otimes [y_p, \dots, \widehat{y_i}, \dots, y_n] \\
&\qquad + \sum_{p=i}^{n} [x_0, \dots, \widehat{x_i}, \dots, x_p] \otimes [y_p, \dots, y_n] \Bigg) \\
&= \sum_{i=0}^{n} \mathrm{AW}([t_0, \dots, \widehat{t_i}, \dots, t_n]) \\
&= \mathrm{AW}(\partial(\sigma)).
\end{aligned}$$

The hard part is to prove that the Alexander-Whitney map induces homology isomorphism. One can do this by constructing the chain homotopy recursively as in the proof of Theorem 4.2.8 or Lemma 2.3.9. However, let us use this as an excuse to introduce **acyclic models**, a method that will allow us to conclude that maps induce isomorphism in homology group with zero thinking.

Definition 6.6.3 ($\mathbb{M}$-free functors).

Let $\mathcal{C}$ be a category and $\mathbb{M}$ be a set of objects in $\mathcal{C}$. A functor $F : \mathcal{C} \to R$-**Mod** is said to be **$\mathbb{M}$-free** if F takes an object $C \in \mathcal{C}$ to a free R-modules generated by symbols of the form $\coprod_{i=1}^{k} \mathrm{Hom}_{\mathcal{C}}(M_i, C)$, where $M_i \in \mathbb{M}$. In other words

$$C \mapsto R\left\{ \coprod_{i=1}^{k} \mathrm{Hom}_{\mathcal{C}}(M_i, C) \right\} \text{ for some } M_i \in \mathbb{M}$$

Example 6.6.4.

If $\mathbb{M} = \{\Delta^n : n \geq 0\}$ and $\mathcal{C} = \mathbf{Top}$, then the functor $X \mapsto S_n(X)$ is $\mathbb{M}$-free for all n.

The case of our interest is $\mathcal{C} = \mathbf{Top} \times \mathbf{Top}$.

Example 6.6.5.

We define
$$\mathbb{M} = \{(\Delta^p, \Delta^q) : p, q \geq 0\}.$$
Then, the two functors
$$\begin{aligned} S_n(\bullet \times \bullet) : \mathbf{Top} \times \mathbf{Top} &\to R\text{-}\mathbf{Mod} \\ (X, Y) &\mapsto S_n(X \times Y) \\ S_n(\bullet) \otimes_R S_n(\bullet) : \mathbf{Top} \times \mathbf{Top} &\to R\text{-}\mathbf{Mod} \\ (X, Y) &\mapsto (S_*(X) \otimes_R S_*(Y))_n \end{aligned}$$
are both $\mathbb{M}$-free. This is because
$$\begin{aligned} S_n(X \times Y, R) &= R\left\{\mathrm{Hom}_{\mathbf{Top}\times\mathbf{Top}}((\Delta^n, \Delta^n), (X, Y))\right\} \\ (S_*(X, R) \otimes_R S_*(Y, R))_n &= R\left\{\coprod_{p+q=n} \mathrm{Hom}_{\mathbf{Top}\times\mathbf{Top}}((\Delta^p, \Delta^q), (X, Y))\right\}. \end{aligned}$$
Collating these two functors for all n, we get two functors
$$S_*(\bullet \times \bullet),\ S_*(\bullet) \otimes_R S_*(\bullet) : \mathbf{Top} \times \mathbf{Top} \to \mathrm{ch}(R\text{-}\mathbf{Mod}),$$
each of which is a chain complex of $\mathbb{M}$-free functors.

We can make an analogue of surjective map, exactness, and free resolutions for $\mathbb{M}$.

Definition 6.6.6.

Unless otherwise specified, all functors are $\mathcal{C} \to R\text{-}\mathbf{Mod}$.

- A natural transformation $\Theta : F \to G$ is **$\mathbb{M}$-surjective** if $\Theta(M) : F(M) \to G(M)$ is surjective for all $M \in \mathbb{M}$. (We could have defined injective, but we won't use it.)
- A sequence of natural transformations $G \to G' \to G''$ is **$\mathbb{M}$-exact** if $G(M) \to G'(M) \to G''(M)$ is exact for all $M \in \mathbb{M}$.
- An **$\mathbb{M}$-free resolution** of T is a chain complex $F_* : \mathcal{C} \to \mathrm{ch}(R\text{-}\mathbf{Mod})$ and a natural transformation $\varepsilon : H_0(F_*) \to T$ such that for all $M \in \mathbb{M}$, $F_*(M)$ is a free resolution of $T(M)$.

We can also mimic the fundamental theorem of homological algebra in the setting. This leads to the following theorem.

Theorem 6.6.7 (Main Theorem of Acyclic Models)**.**

Let $\Theta : S \to T$ be a natural transformation of functors $S, T : \mathcal{C} \to R\text{-}\mathbf{Mod}$. If F_* and G_* are $\mathbb{M}$-free resolution of S and T, then there is a (unique up to natural chain homotopy)
$$\Delta_* : F_* \to G_*$$
such that $H_0(\Delta_*) = \Theta$.

To prove this theorem, it suffices to prove the lifting condition similar to (♠) in the proof of Theorem 6.4.5. In particular, we prove the following lemma.

Lemma 6.6.8 (Lifting condition)**.**

Let $F, P, Q : \mathcal{C} \to R\text{-}\mathbf{Mod}$ be functors. Suppose that F is $\mathbb{M}$-free and $\Theta : P \to Q$ is a $\mathbb{M}$-surjective natural transformation. Let $f : F \to Q$ be any natural transformation. Then there is a lifting $\overline{f} : F \to P$.

$$\begin{tikzcd} & F \arrow[dl, dashed, "\exists \overline{f}"'] \arrow[d, "f"] \\ P \arrow[r, two heads, "\Theta"'] & Q \end{tikzcd}.$$

▶ *Proof.* Clearly, we may assume that $F(X) = R\{\mathrm{Hom}_{\mathcal{C}}(M, X)\}$ for some $M \in \mathbb{M}$. We first define the lifting $\overline{f}_M : F(M) \to P(M)$. Let $1_M \in R\{\mathrm{Hom}_{\mathcal{C}}(M, M)\}$. By surjectivity, $f_M(1_M) \in Q(M)$ is hit by some element $c_M \in P(M)$. We define $\overline{f}(1_M) = c_M$.

Now, we exploit naturality to define $\overline{f}_X \to F(X) \to P(X)$ for all $X \in \mathcal{C}$. For any $\varphi : M \to X$, chasing the diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(M, M) \arrow[r, "\overline{f}_M"] \arrow[d, "\varphi^*"] & P(M) \arrow[d, "P(\varphi)"] \\ \mathrm{Hom}_{\mathcal{C}}(M, X) \arrow[r, "\overline{f}_X"] & P(X) \end{tikzcd}$$

should convince you that you must pick $\overline{f}_X(\varphi) = \varphi_*(c_M)$. It's easy to check that this work. □

Theorem 6.6.7 follows from the previous lemma by following the proof of Fundamental Theorem of Homological Algebra in verbatim. We leave the detail to the reader.

Finally, we prove Eilenberg-Zilber Theorem.

▶ *Proof of Eilenberg-Zilber Theorem (Theorem 6.6.1).* We apply Theorem 6.6.7 in the specific case of Example 6.6.5. Note that for any $p, q \geq 0$, the chain complexes $S_*(\Delta^p \times \Delta^q, R)$ and $S_*(\Delta^p, R) \otimes_R S_*(\Delta^q, R)$ are free resolutions of $H_0(\Delta^p \times \Delta^q, R)$ and $H_0(\Delta^p, R) \otimes_R H_0(\Delta^q, R)$, respectively (this uses the fact that Δ^p, Δ^q, and $\Delta^p \times \Delta^q$ are contractible.) This implies by definition that $S_*(\bullet \times \bullet)$ and $S_*(\bullet) \otimes_R S_*(\bullet)$ are free resolutions of $H_0(\bullet \times \bullet, R)$ and $H_0(\bullet, R) \otimes_R H_0(\bullet, R)$, respectively. Therefore, the Alexander Whitney map

$$\mathrm{AW} : S_*(X \times Y, R) \to S_*(X, R) \otimes_R S_*(Y, R)$$

is a natural transformation lifting

$$H_0(\mathrm{AW}) : H_0(X \times Y, R) \to H_0(X, R) \otimes_R H_0(Y, R),$$

which is clearly an isomorphism.

By the theorem above, there is a chain homotopy between functors $S_*(X \times Y, R)$ and $S_*(X, R) \otimes_R S_*(Y, R)$. This makes the homology groups of these two chain complexes isomorphic. □

To make it explicit, Eilenberg-Zilber theorem implies that

Corollary 6.6.9 (Kunneth Formula)**.**

If R is a PID, then

$$H_n(X \times Y, R) = \bigoplus_{p+q=n} H_p(X, R) \otimes_R H_q(Y, R) \oplus \bigoplus_{p+q=n-1} \mathrm{Tor}_1^R(H_p(X, R), H_q(Y, R)).$$

> This isomorphism is not natural. However, if R is a field, then the second big direct sum is gone, and the isomorphism is natural.

§6.7 Problems

Problem 6.A. Mark has a necklace consisting of beads of n different colors arranged in a line such that the number of beads that have the i-th color is $2a_i$ for all $1 \le i \le n$. Prove that Mark can split the necklace using at most n cuts and reassemble them into two pieces so that both pieces have exactly a_i beads that have the i-th color for all $1 \le i \le n$.

Problem 6.B. Let $R = \mathbb{C}[x, y]/(xy)$. Compute $\operatorname{Tor}_n^R(R/(x), R/(x))$. Your answer should depend on whether $n = 0$, n is odd, or n is even and positive.

Problem 6.C. For each prime p, construct two maps $f, g : X \to Y$ between two topological spaces X and Y such that $H_n(f, \mathbb{Z}) = H_n(g, \mathbb{Z})$ for all n, but there exists n such that $H_n(f, \mathbb{F}_p) \neq H_n(g, \mathbb{F}_p)$.

Problem 6.D. Let K be the Klein bottle.

(a) Compute $H_n(K \times K, \mathbb{Z})$ for any n.

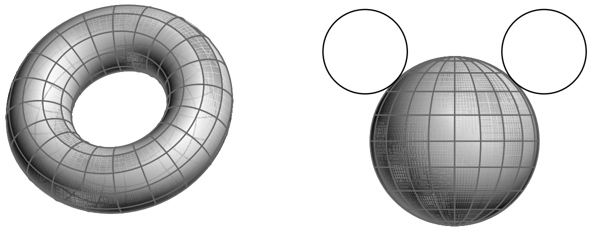
(b) Compute $H_n(K \times K, \mathbb{F}_2)$ for any n.

7 Cohomology

For the past three chapters, we have seen that homology is a useful invariant that captures the concept of n-dimensional holes. However, homology is far from a perfect invariant that distinguishes any pair of topological spaces up to homotopy equivalence. It is very easy to construct two topological spaces with the same homology groups.

Example 7.0.1.

Consider the torus $\Sigma = S^1 \times S^1$ and the *mickey mouse space* $M = S^1 \vee S^2 \vee S^1$.



Σ $\qquad$ M

We can easily check the homology groups of both spaces are $H_0(\Sigma) = H_0(M) = \mathbb{Z}$, $H_1(\Sigma) = H_1(M) = \mathbb{Z}^2$, and $H_2(\Sigma) = H_2(M) = \mathbb{Z}$.

An observant reader might notice that the fundamental group of these two spaces are not equal. In particular, $\pi_1(\Sigma) = \mathbb{Z}^2$ but $\pi_1(M) = \langle a, b\rangle$. Thus, Σ and M are not homotopy equivalent. In this chapter, we will provide a different method to distinguish Σ and M.

(If you are still bothered by the fact that these can be distinguished by π_1, you can take $S^2 \times S^2$ and $S^2 \vee S^2 \vee S^4$ instead.)

To distinguish spaces like Σ and M, we note that for any topological space X, there is the **diagonal map**

$$\begin{aligned} \Delta : X &\to X \times X \\ x &\mapsto (x, x). \end{aligned}$$

This diagonal map provides additional information about the topological space. In particular, taking H_n (over $\mathbb{F}_2$, say) yields the map

$$H_n(\Delta, \mathbb{F}_2) : H_n(X, \mathbb{F}_2) \to H_n(X \times X, \mathbb{F}_2) \simeq \bigoplus_{p+q=n} H_p(X, \mathbb{F}_2) \otimes_{\mathbb{F}_2} H_q(X, \mathbb{F}_2),$$

where the isomorphism on the right is provided by Eilenberg-Zilber (Corollary 6.6.9).

More concisely, if we define the graded $\mathbb{F}_2$-vector space

$$H_*(X, \mathbb{F}_2) := \bigoplus_{n \geq 0} H_n(X, \mathbb{F}_2).$$

Then the above yields a map

$$H_*(\Delta, \mathbb{F}_2) : H_*(X, \mathbb{F}_2) \to H_*(X, \mathbb{F}_2) \otimes_{\mathbb{F}_2} H_*(X, \mathbb{F}_2).$$

It turns out that in both examples above (Σ versus M and $\mathbb{CP}^4$ versus $S^2 \vee S^4$), the structure of the map $H_*(\Delta, \mathbb{F}_2)$ can distinguish both spaces.

Example 7.0.2.

Suppose that $H_0(\Sigma, \mathbb{F}_2)$ is generated by p, $H_1(\Sigma, \mathbb{F}_2)$ is generated by a and b, and $H_2(\Sigma, \mathbb{F}_2)$ is generated by x. Then the map $H_*(\Delta, \mathbb{F}_2)$ (for Σ) sends

$$x \mapsto p \otimes x + a \otimes b + x \otimes p.$$

In contrast, suppose instead that $H_0(M, \mathbb{F}_2)$ is generated by p, $H_1(M, \mathbb{F}_2)$ is generated by a and b, and $H_2(M, \mathbb{F}_2)$ is generated by x. Then $H_*(\Delta, \mathbb{F}_2)$ (for M) sends

$$x \mapsto p \otimes x + x \otimes p,$$

making those two maps different.

We will not take time to justify these two claims.

We now have an invariant that can distinguish more pairs of topological spaces. However, this invariant is not super pleasant to work with because it is a map from $M \to M \otimes M$. In other words, it is a **comultiplication** structure.

If we instead have a topological invariant similar to homology but is a contravariant functor, say H^*. Then the diagonal map would induce a map $H^*(X \times X, \mathbb{F}_2) \to H^*(X, \mathbb{F}_2)$. Then one might hope to turn this into a map $H^*(X, \mathbb{F}_2) \otimes_{\mathbb{F}_2} H^*(X, \mathbb{F}_2) \to H^*(X, \mathbb{F}_2)$, which actually gives a multiplication structure. This will promote $H^*(X, \mathbb{F}_2)$ into the **cohomology ring**, which encodes the structure of the diagonal map.

But how do we get such a contravariant functor? The answer is we dualize everything. As a simpler analogy, for any linear map $\phi : V \to W$ of k-vector spaces, we have the induced linear map $W^\vee \to V^\vee$ by sending $f : W \to k$ to $f \circ \phi$. We will apply this dualizing construction to the homology, giving another invariant called the **cohomology** $H^n(X)$. The construction will be detailed in Section 7.1.

In the following Section 7.2, we show that cohomology is completely determined by homology by giving a variant of the Universal Coefficient Theorem. Although this makes cohomology not an interesting invarianta on its own, the preceding discussion hints at how to promote cohomology into a ring. More specifically, we will describe the multiplication opearation called the **cup product**, which takes an $x \in H^p(X)$ and $y \in H^q(X)$ and produces $xy \in H^{p+q}(X)$. We explain this in Section 7.3

§7.1 Cohomology

We now setup the cohomology.

§7.1.1 Setup

We now define singular cohomology. Let us first start with absolute cohomology. We first explain how to dualize the chain complex.

Definition 7.1.1 (Dual Chain Complex).

For any chain complex C_* and abelian group M, we have the **dual chain complex** $\operatorname{Hom}(C_*, M)$ whose entry at $-n$ is

$$(\operatorname{Hom}(C_*, M))_{-n} = \operatorname{Hom}_{\mathbb{Z}}(C_n, M),$$

which comes with boundary map

$$\begin{aligned}\partial : \operatorname{Hom}(C_n, M) &\to \operatorname{Hom}(C_{n+1}, M)\\ (\partial f)(\sigma) &= (-1)^{n+1} f(\partial\sigma).\end{aligned}$$

(Check yourself that $\partial^2 = 0$.)

Note that any map $f : C_* \to D_*$ induces a map $f^* : \operatorname{Hom}(D_*, M) \to \operatorname{Hom}(C_*, M)$ in an obvious way.

Definition 7.1.2 (Singular Cohomology)**.**

For any topological space X, let

$$S^*(X, M) = \operatorname{Hom}(S_*(X), M),$$

and the **singular cohomology** of X is given by

$$H^n(X, M) = H_{-n}(S^*(X, M)).$$

In particular, one should think of an element $S^n(X, M)$ as a function that takes an n-chain to an element in M.

Warning 7.1.3.

In the definition of ∂ above, [Mil21] has $(-1)^{n+1}$ sign, but [Hat02] does not. These two ways of defining $S^*(X, M)$ will result in isomorphic cohomology groups, so it does not affect what follows.

Next, we define relative cohomology. For any subspace $A \subseteq X$, we note that $S^n(A, M)$ is a quotient of $S^n(X, M)$. In particular, there is a surjective map

$$S^n(X, M) \to S^n(A, M)$$

obtained by restricting any map $f : S_n(X) \to M$ to $S_n(A)$. This map has a kernel

$$S^n(X, A, M) := \{f : S_n(X) \to M \text{ such that } f(\sigma) = 0 \text{ for all } \sigma \in S_n(A)\},$$

and the map ∂ promotes this to a complex $S^*(X, A, M)$, where $S^n(X, A, M)$ takes degree $-n$.

Definition 7.1.4 (Relative Cohomology)**.**

For any subspace $A \subseteq X$ and abelian group M, we define the **relative cohomology group**

$$H^n(X, A, M) = H_{-n}(S^*(X, A, M)).$$

§7.1.2 Computation Examples

Example 7.1.5.

Let us compute the cohomology of $\mathbb{RP}^2$. The cell structure $C_*^{\text{cell}}(\mathbb{RP}^2, \mathbb{Z})$ is given by

$$\cdots \longrightarrow 0 \longrightarrow \underset{\deg 2}{\mathbb{Z}} \xrightarrow{\cdot 2} \underset{\deg 1}{\mathbb{Z}} \xrightarrow{0} \underset{\deg 0}{\mathbb{Z}} \longrightarrow 0 \longrightarrow \cdots.$$

One can compute the cohomology group by taking dual of the above chain complex, giving the complex $C^*_{\text{cell}}(\mathbb{RP}^2, \mathbb{Z})$

$$\dots \longrightarrow 0 \longrightarrow \underset{\deg 0}{\mathbb{Z}} \xrightarrow{0} \underset{\deg -1}{\mathbb{Z}} \xrightarrow{\cdot 2} \underset{\deg -2}{\mathbb{Z}} \longrightarrow 0 \longrightarrow \dots,$$

and then one can read off the cohomology groups

$$\begin{aligned} H^0(\mathbb{RP}^2, \mathbb{Z}) &= \mathbb{Z} \\ H^1(\mathbb{RP}^2, \mathbb{Z}) &= 0 \\ H^2(\mathbb{RP}^2, \mathbb{Z}) &= \mathbb{Z}/2\mathbb{Z} \\ H^n(\mathbb{RP}^2, \mathbb{Z}) &= 0 \qquad \text{for all } n \geq 3. \end{aligned}$$

Let us do the same thing with $\mathbb{F}_2$.

Example 7.1.6.

One can compute the cohomology group on $\mathbb{F}_2$ by taking dual from the chain complex in $\mathbb{Z}$ to $\mathbb{F}_2$. In particular,

$$C^i_{\text{cell}}(\mathbb{RP}^2, \mathbb{F}_2) = \operatorname{Hom}(C_i^{\text{cell}}(\mathbb{RP}^2, \mathbb{Z}), \mathbb{F}_2).$$

The result of this gives the complex $C^*_{\text{cell}}(\mathbb{RP}^2, \mathbb{F}_2)$

$$\dots \longrightarrow 0 \longrightarrow \underset{\deg 0}{\mathbb{F}_2} \xrightarrow{0} \underset{\deg -1}{\mathbb{F}_2} \xrightarrow{\cdot 2} \underset{\deg -2}{\mathbb{F}_2} \longrightarrow 0 \longrightarrow \dots,$$

and then one can read off the cohomology groups

$$\begin{aligned} H^0(\mathbb{RP}^2, \mathbb{F}_2) &= \mathbb{F}_2 \\ H^1(\mathbb{RP}^2, \mathbb{F}_2) &= \mathbb{F}_2 \\ H^2(\mathbb{RP}^2, \mathbb{F}_2) &= \mathbb{F}_2 \\ H^n(\mathbb{RP}^2, \mathbb{F}_2) &= 0 \qquad \text{for all } n \geq 3. \end{aligned}$$

§7.1.3 Basic Properties of Cohomology

We now collect some basic facts about cohomology. We will not prove most of these since (i) we will not use it very much, since later, we will see that cohomology is determined by homology, and (ii) most of them are straightforward to verify.

► **0-th cohomology.** Recall that $H_0(X, R)$ is the free R-module generated by the path components of X. Similarly, $H^0(X, R)$ is an R-module of functions from path components to R (it's just looks like the dual). In other words,

$$H^0(X, R) = \operatorname{Hom}_R(R\{\pi_0 X\}, R) = \operatorname{Hom}_R(H_0(X, R), R).$$

► **Cohomology is a functor.** If $f : X \to Y$ is a continuous map, then there are induced map of R-modules

$$H^q(Y, R) \to H^q(X, R).$$

In particular, for every q, $H^q(-, R)$ is a functor from $\mathbf{Top}^{\text{op}} \to R\text{-}\mathbf{Mod}$.

Moreover, this functor is homotopy invariant: if $f, g : X \to Y$ are homotopic, then the maps $H^q(f, R)$ and $H^q(g, R)$ are the same.

▶ **The Long Exact Sequence of Cohomology** For any subspace $A \subseteq X$, the short exact sequence

$$0 \longrightarrow S^*(X,A,M) \longrightarrow S^n(X,M) \longrightarrow S^n(A,M) \longrightarrow 0$$

induces (by Theorem 4.3.11) the long exact sequence

$$\begin{tikzcd}
0 \arrow[r] & H^0(X,A,M) \arrow[r] & H^0(X,M) \arrow[r] & H^0(A,M) \arrow[dll] \\
 & H^1(X,A,M) \arrow[r] & H^1(X,M) \arrow[r] & H^1(A,M) \arrow[dll] \\
 & H^2(X,A,M) \arrow[r] & H^2(X,M) \arrow[r] & H^2(A,M) \arrow[r] & \dots
\end{tikzcd}$$

▶ **Excision** Excision still works. In particular, if $U \subseteq A \subseteq X$ is such that U is an open subset of A, and the closure of U is contained in the interior of A, then similar to excision Theorem 4.4.1, we get that the induced map

$$H^n(X,A) \to H^n(X-U, A-U)$$

is an isomorphism.

Similarly, if $A \subset X$ is reasonable in the sense of the statement of Theorem 4.4.2, then $H^n(X,A) \simeq H^n(X/A, \{\mathrm{pt}\})$.

§7.2 Ext and Universal Coefficient Theorem

In order to get a universal coefficient theorem for cohomology, we need to dualize the Tor functor. The result is the Ext functor.

Definition 7.2.1.

Given R-modules M and N, we define $\mathrm{Ext}^i_R(M,N)$ as follows: take a free resolution F_* of M, and let

$$\mathrm{Ext}^i_R(M,N) = H_{-i}(\mathrm{Hom}_R(F_*,N)).$$

Note that $\mathrm{Ext}^0_R(M,N) = \mathrm{Hom}_R(M,N)$.

Using the fundamental theorem of homological algebra Theorem 6.4.5 in the same way as proving that Tor does not depend on free resolution, we can prove that the above definiton does not depend on the free resolution.

Proposition 7.2.2 (Properties of Ext)**.**

(a) We have

$$\mathrm{Ext}^i_R(M \oplus M', N) \simeq \mathrm{Ext}^i_R(M,N) \oplus \mathrm{Ext}^i_R(M',N)$$
$$\mathrm{Ext}^i_R(M, N \oplus N') \simeq \mathrm{Ext}^i_R(M,N) \oplus \mathrm{Ext}^i_R(M,N')$$

(b) Given a short exact sequence $0 \to A \to B \to C \to 0$ of R-modules, we have a long exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_R(M,A) \arrow[r] & \mathrm{Hom}_R(M,B) \arrow[r] & \mathrm{Hom}_R(M,C) \arrow[dll] \\
& \mathrm{Ext}^1_R(M,A) \arrow[r] & \mathrm{Ext}^1_R(M,B) \arrow[r] & \mathrm{Ext}^1_R(M,C) \arrow[dll] \\
& \mathrm{Ext}^2_R(M,A) \arrow[r] & \mathrm{Ext}^2_R(M,B) \arrow[r] & \mathrm{Ext}^2_R(M,C) \arrow[r] & \dots
\end{tikzcd}$$

▶ *Proof.* (a) The first equation follows from that if F_* is a free resolution of M and F'_* is a free resolution of M', then $F_* \oplus F'_*$ is a free resolution of $M \oplus M'$. The second equations follows directly from the fact that $\mathrm{Hom}_R(P, N \oplus N') \simeq \mathrm{Hom}_R(P,N) \oplus \mathrm{Hom}_R(P,N')$.

(b) Same proof as Proposition 6.4.11 (c). In particular, we have the short exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(F_*, A) \longrightarrow \mathrm{Hom}_R(F_*, B) \longrightarrow \mathrm{Hom}_R(F_*, C) \longrightarrow 0,$$

which induces the desired long exact sequence by Theorem 4.3.11. □

Example 7.2.3 (Example computation of Ext)**.**

We have

$$\begin{aligned}
\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/2, \mathbb{Z}/4) &= H_{-1}\left(\mathrm{Hom}_{\mathbb{Z}}\left(\underset{\deg 1}{\mathbb{Z}} \xrightarrow{2} \underset{\deg 0}{\mathbb{Z}}, \underset{\deg 0}{\mathbb{Z}/4}\right)\right) \\
&= H_{-1}\left(0 \longrightarrow \underset{\deg 0}{\mathbb{Z}/4} \xrightarrow{2} \underset{\deg -1}{\mathbb{Z}/4} \longrightarrow 0\right) \\
&= \mathbb{Z}/2.
\end{aligned}$$

By computing every possible cases, we have that

Proposition 7.2.4.

We have

$$\mathrm{Ext}^i_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) = \begin{cases} \mathbb{Z}/\gcd(m,n)\mathbb{Z} & \text{if } i = 0, 1 \\ 0 & \text{if } i \geq 2. \end{cases}$$

$$\mathrm{Ext}^i_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) = \begin{cases} \mathbb{Z}/n\mathbb{Z} & \text{if } i = 0 \\ 0 & \text{if } i \geq 1. \end{cases}$$

$$\mathrm{Ext}^i_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) = \begin{cases} 0 & \text{if } i = 0 \text{ or } i \geq 2 \\ \mathbb{Z}/n\mathbb{Z} & \text{if } i = 1. \end{cases}$$

Warning 7.2.5.

Unlike Tor, the Ext functor is not symmetric. Even worse, you **cannot** compute $\mathrm{Ext}^i_R(M,N)$ by taking a free resolution of N.

(In fact, reducing N is possible, but you have to use another kind of resolution called the **injective resolution** of N to get the correct result.)

We now state the universal coefficient theorem for cohomology.

Theorem 7.2.6 (Universal Coefficient Theorem)**.**

If X is a topological space and R is a commutative ring, then

$$H^n(X,R) \simeq \operatorname{Hom}_{\mathbb{Z}}(H_n(X),R) \oplus \operatorname{Ext}^1_{\mathbb{Z}}(H_{n-1}(X),R).$$

▶ *Proof.* Similar to the Universal Coefficient Theorem for homology (Theorem 6.5.1). Either proof could be easily adapted to prove this theorem. □

Example 7.2.7.

We have

$$\begin{aligned} H^2(\mathbb{RP}^2,\mathbb{F}_2) &\simeq \operatorname{Hom}_{\mathbb{Z}}(H_2(\mathbb{RP}^2),\mathbb{F}_2) \oplus \operatorname{Ext}^1_{\mathbb{Z}}(H_1(\mathbb{RP}^2),\mathbb{F}_2) \\ &\simeq \operatorname{Hom}_{\mathbb{Z}}(0,\mathbb{F}_2) \oplus \operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{F}_2,\mathbb{F}_2) \\ &\simeq 0 \oplus \mathbb{F}_2 \simeq \mathbb{F}_2 \\ H^1(\mathbb{RP}^2,\mathbb{F}_2) &\simeq \operatorname{Hom}_{\mathbb{Z}}(H_1(\mathbb{RP}^2),\mathbb{F}_2) \oplus \operatorname{Ext}^1_{\mathbb{Z}}(H_0(\mathbb{RP}^2),\mathbb{F}_2) \\ &\simeq \operatorname{Hom}_{\mathbb{Z}}(\mathbb{F}_2,\mathbb{F}_2) \oplus \operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Z},\mathbb{F}_2) \\ &\simeq \mathbb{F}_2 \oplus 0 \simeq \mathbb{F}_2. \end{aligned}$$

§7.3 Cohomology Ring

As mentioned earlier, the main purpose of cohomology is to turn comultiplication structure into multiplication structure by flipping the arrow. For any ring R, we will define a multiplication structure on

$$H^*(X,R) = \bigoplus_{n\geq 0} H^n(X,R),$$

which will promote $H^*(X,R)$ to a ring.

§7.3.1 Definition and Basic Properties of Cup Product

To do this, let us first think about what we want. We want to define a map

$$H^*(X,R) \otimes H^*(X,R) \longrightarrow H^*(X,R),$$

using the map $H^*(\Delta,R)$, where $\Delta : X \to X \times X$ is the diagonal map. Therefore, we want to be able to make a composition

$$H^*(X,R) \otimes H^*(X,R) \longrightarrow H^*(X \times X,R) \overset{H^*(\Delta,R)}{\longrightarrow} H^*(X,R),$$

and so it suffices to define the first map. Intuitively, the first map should be the dual of the corresponding homology map $H_*(X \times X,R) \to H_*(X,R) \times H_*(X,R)$, which is the Alexander-Whitney map!

We now begin constructing this map.

Definition 7.3.1 (Cohomology Cross Product)**.**

For any topological space X and Y, we define the **cohomology cross product**

$$\times : H^*(X,R) \otimes H^*(Y,R) \to H^*(X \times Y,R)$$

in three steps:

$$
\begin{tikzcd}
H^*(X,R) \otimes_R H^*(Y,R) \arrow[d, "\mathrm{UCT}"] & \\
H^*(S^*(X,R) \otimes_R S^*(Y,R)) \arrow[r, dash, dashed] \arrow[dd, dashed] & S^*(X,R) \otimes_R S^*(Y,R) \arrow[d] \\
& \operatorname{Hom}(S_*(X,R) \otimes_R S_*(Y,R), R) \arrow[d, "\mathrm{AW}^*"] \\
H^*(X \times Y, R) \arrow[r, dash, dashed] & \operatorname{Hom}(S_*(X \times Y, R), R) = S^*(X \times Y, R)
\end{tikzcd}
$$

To explain the diagram above, the dashed arrow in the left column is obtained by taking homology from the right column. We now explain each map.

- The topmost map is obtained from the universal coefficient theorem.
- The middle map is the nonobvious one. For any $f \in S^p(X,R)$ and $g \in S^q(Y,R)$, the map takes (f,g) to $h : S_*(X,R) \otimes_R S_*(Y,R) \to R$ defined by

  $$h(\alpha \otimes \beta) = (-1)^{pq} f(\alpha) g(\beta)$$

  for all $\alpha \in S_p(X,R)$ and $\beta \in S_q(X,R)$.
- The bottom map is induced from the Alexander-Whitney map $\mathrm{AW} : S_*(X \times Y, R) \to S_*(X,R) \otimes S_*(Y,R)$.

To make sure that this definition is valid, we have to check that the middle map is indeed a chain map. Again, this is a tedious calculation.

Cross product does not necessarily induce isomorphism at the homology level. However, it does in many cases, such as when the conditions of the following proposition hold.

Proposition 7.3.2.

If R is a PID and $H_*(X,R)$ is a finitely-generated free R-module, then the cross product

$$\times : H^*(X,R) \otimes H^*(Y,R) \to H^*(X \times Y, R)$$

is an isomorphism.

▶ *Proof.* Omitted. See [Mil21, Thm. 33.3]. The main point is to show, using UCT, that the middle map induces isomorphism in homology. □

We now define the cup product.

Definition 7.3.3 (Cup Product)**.**

For any topological space X, the **cup product** is a map

$$\smile : H^*(X,R) \otimes_R H^*(X,R) \to H^*(X,R)$$

defined by the composition of cross product and H^* of the diagonal map $\Delta : X \to X \times X$.

$$H^*(X,R) \otimes_R H^*(X,R) \xrightarrow{\times} H^*(X \times X, R) \xrightarrow{H^*(\Delta)} H^*(X,R).$$

Let us understand what is going on with this map. Suppose that $\overline{f} \in H^p(X,R)$ is represented by $f : S_p(X,R) \to R$ and $\overline{g} \in H^q(X,R)$ is represented by $g : S_q(X,R) \to R$. Then if one stares at the above construction long enough, one would find that $\overline{f} \smile \overline{g} \in H^{p+q}(X, R)$ is represented by $f \smile g \in S_n(X, R) \to R$ defined by

$$\boxed{(f \smile g)(\sigma) = (-1)^{pq}\, f(\sigma|_{\Delta^p,\text{front}})\, g(\sigma|_{\Delta^q,\text{back}})} \tag{7.1}$$

for all $\sigma : \Delta^{p+q} \to X$. Roughly speaking, the $(-1)^{pq}$ and multiplication between f and g comes from the middle map, while the restrictions to front and back faces come from Alexander-Whitney map.

Here $\sigma|_{\Delta^p,\text{front}}$ denotes the front p-face of σ (i.e., take the first $p+1$ vertices of σ). Similarly, $\sigma|_{\Delta^q,\text{back}}$ denotes the back q-face of σ (i.e., take the last $q+1$ vertices of σ).

We now prove some properties of cohomology rings. This is where acyclic model that we develop when we proved Eilenberg-Zilber theorem will shine.

Theorem 7.3.4 (Properties of Cohomology Ring)**.**

Let X be a topological space and R be a ring. Then, the cup product satisfies

(a) it is **graded**, i.e., if $\overline{f} \in H^p(X, R)$ and $\overline{g} \in H^q(X, R)$, then $\overline{f} \smile \overline{g} \in H^{p+q}(X, R)$.

(b) it has **multiplicative identity** $\mathbf{1} \in H^0(X, R)$ represented by the function taking each point in X to $1 \in R$.

(c) it is **associative**; $(\overline{f} \smile \overline{g}) \smile \overline{h} = \overline{f} \smile (\overline{g} \smile \overline{h})$.

(d) it is **graded commutative**: if $\overline{f} \in H^p(X, R)$ and $\overline{g} \in H^q(X, R)$, then

$$(\overline{f} \smile \overline{g}) = (-1)^{pq}(\overline{g} \smile \overline{f}).$$

The fact that cup product is only commutative up to signs makes cohomology rings with coefficient being arbitrary ring annoying. To avoid sign issues, algebraic topologist often work in $\mathbb{F}_2$, in which there is no sign issue, and the cohomology ring is always a commutative ring.

▶ *Proof.* (a) Clear.

(b) Clear.

(c) Let us give two proofs for this.

The first proof is a direct calculation, which shows that associativity is true at the chain level (not just homology level). Suppose that

$$\overline{f} \in H^p(X, R), \quad \overline{g} \in H^q(X, R), \quad \overline{h} \in H^r(X, R),$$

where $\overline{f}$ is represented by $f : S_p(X, R) \to R$ and similarly for $\overline{g}, g$ and $\overline{h}, h$. Then, for any $\sigma \in \text{Sing}_{p+q+r}(X)$, we compute

$$\begin{aligned} \big((f \smile g) \smile h\big)(\sigma) &= (-1)^{(p+q)r}\, (f \smile g)(\sigma|_{\Delta^{p+q},\text{front}})\, h(\sigma|_{\Delta^r,\text{back}}) \\ &= (-1)^{pr+qr}(-1)^{pq}\, f(\sigma|_{\Delta^p,\text{front}})\, g(\sigma|_{\Delta^q,\text{middle}})\, h(\sigma|_{\Delta^r,\text{back}}) \\ &= (-1)^{pq+qr+pr}\, f(\sigma|_{\Delta^p,\text{front}})\, g(\sigma|_{\Delta^q,\text{middle}})\, h(\sigma|_{\Delta^r,\text{back}}), \end{aligned}$$

where $\sigma|_{\Delta^q,\text{middle}}$ denotes the restriction of σ to the face consisting of vertices indexed $p, p+1, \ldots, p+q$.

Analogously, we have

$$\big(f \smile (g \smile h)\big)(\sigma) = (-1)^{pq+qr+pr}\, f(\sigma|_{\Delta^p,\text{front}})\, g(\sigma|_{\Delta^q,\text{middle}})\, h(\sigma|_{\Delta^r,\text{back}}),$$

proving associativity.

The second proof overkills this with acyclic model (as in the proof of Eilenberg-Zilber). Note that it suffices to prove that cross product is associative, since the diagonal map is clearly associative. To do this, consider two chain maps

$$S^*(X, R) \otimes_R S^*(Y, R) \otimes_R S^*(Z, R) \longrightarrow S^*(X \times Y \times Z, R)$$

defined by $x \otimes y \otimes z$ maps to $(x \times y) \times z$ and $x \times (y \times z)$, respectively. Observe that

- The functors $(X, Y, Z) \mapsto S^*(X, R) \otimes_R S^*(Y, R) \otimes_R S^*(Z, R)$ and $(X, Y, Z) \mapsto S^*(X \times Y \times Z, R)$ are $\mathbb{M}$-free, where $\mathbb{M} = \{(\Delta^p, \Delta^q, \Delta^r) : p, q, r \geq 0\}$.
- Both maps above are $\mathbb{M}$-exact because Δ^p, Δ^q, Δ^r, and Δ^{p+q+r} are contractible.
- Both maps induces identity map at H_0.

Therefore, by Theorem 6.6.7, these two maps are chain homotopic, so $(x \times y) \times z$ and $x \times (y \times z)$ at the homology level.

(d) Unlike associativity, this one is no longer true at the chain level. Thus, the direct calculation proof would have to establish some chain homotopy. Therefore, we will use acyclic models.

If one looks at all the maps carefully, the UCT map and the middle map are clearly commutative. Thus, it suffices to show that the Alexander-Whitney map is graded commutative. More precisely, define the **signed-swap map**

$$\begin{aligned} T : S_*(X, R) \otimes_R S_*(Y, R) &\to S_*(Y, R) \otimes_R S_*(X, R) \\ T(\alpha \otimes \beta) &= (-1)^{pq}(\beta \otimes \alpha) \quad \text{for all } \alpha \in S_p(X, R) \text{ and } \beta \in S_q(Y, R). \end{aligned}$$

Also, define the map $s : X \times Y \to Y \times X$ by $(a, b) \mapsto (b, a)$ for all $a \in X$ and $b \in Y$. Then we claim that the following diagram commutes up to chain homotopy:

$$\begin{tikzcd} S_*(X \times Y, R) \arrow[r, "\mathrm{AW}"] \arrow[d, "s_*"] & S_*(X, R) \otimes_R S_*(Y, R) \\ S_*(Y \times X, R) \arrow[r, "\mathrm{AW}"] & S_*(Y, R) \otimes_R S_*(X, R). \arrow[u, "T"'] \end{tikzcd} \tag{7.2}$$

To show this, we first show that the bottom path is a chain map (i.e., commute with ∂). In particular, it suffices to show that T is a chain map. To do this, for any $\alpha \in S_p(X, R)$ and $\beta \in S_q(Y, R)$, we compute

$$\begin{aligned} T(\partial(\alpha \otimes \beta)) &= T(\partial\alpha \otimes \beta + (-1)^p(\alpha \otimes \partial\beta)) \\ &= (-1)^{(p-1)q}(\beta \otimes \partial\alpha) + (-1)^{p(q-1)} \cdot (-1)^p(\partial\beta \otimes \alpha) \\ &= (-1)^{pq}(\partial\beta \otimes \alpha) + (-1)^{pq+q}(\beta \otimes \partial\alpha) \\ &= (-1)^{pq}\, \partial(\beta \otimes \alpha) \\ &= \partial(T(\beta \otimes \alpha)). \end{aligned}$$

(Once again, notice that $(-1)^{pq}$ is crucial for the calculation.) Now, we use acyclic model.

- The functors $(X, Y) \mapsto S_*(X, R) \otimes S^*(Y, R)$ and $(X, Y) \mapsto S_*(X \times Y, R)$ is $\mathbb{M}$-free, where $\mathbb{M} = \{(\Delta^p, \Delta^q) : p, q \geq 0\}$.
- Both natural transformations AW and $T \circ \mathrm{AW} \circ s_*$ are $\mathbb{M}$-exact because Δ^p and $\Delta^p \times \Delta^q$ are both contractible.
- Both maps induces the identity map at H_0.

Thus, the main theorem of acyclic models concludes (Theorem 6.6.7) gives that AW and $T \circ \mathrm{AW} \circ s_*$ are chain homotopic, justifying the diagram (7.2).

We specialize to $Y = X$. Finally, taking the dual of (7.2) gives the diagram

$$\begin{tikzcd} \mathrm{Hom}(S_*(X, R) \otimes_R S_*(X, R), R) \arrow[r, "\mathrm{AW}^*"] \arrow[d, "s^*"] & S^*(X \times X, R) \\ \mathrm{Hom}(S_*(X, R) \otimes_R S_*(X, R), R) \arrow[r, "\mathrm{AW}^*"] & S^*(X \times X, R) \arrow[u, "T^*"'] \end{tikzcd},$$

which show that the AW step in the definition of cohomology cross product is graded commutative. In particular, $x \times y$ and $(-1)^{pq}(y \times x)$ are the same at the homology level. (The $(-1)^{pq}$ comes from the definition of T.) Composing this with the diagonal map gives that cup product is graded commutative. $\square$

§7.3.2 Examples

We now try to compute cohomology ring for spaces that we can compute directly.

▶ **Cohomology Ring of Sphere.** We compute $H^*(S^n, \mathbb{Z})$.

First, by universal coefficient theorem or by dualizing your favorite complex, we have that

$$H^i(S^n, \mathbb{Z}) = \begin{cases} \mathbb{Z} & i = 0, n \\ 0 & i \neq 0, n. \end{cases}$$

Therefore, we get the module structure

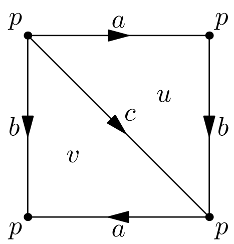
$$H^*(S^n, \mathbb{F}_2) = \mathbb{Z}_2\{1, \omega\},$$

where $\deg 1 = 0$ and $\deg \omega = n$. For degree reasons, we have $\omega^2 = 0$, so the cohomology ring is

$$\boxed{H^*(S^n, \mathbb{Z}_2) \simeq \frac{\mathbb{Z}[\omega]}{(\omega^2)} \qquad \text{where } \deg \omega = n}.$$

▶ **Cohomology Ring of Klein Bottle.** We compute $H^*(K, \mathbb{F}_2)$, where K is the Klein bottle. The purpose of this is to show that cohomology rings can be computed directly by decomposing a space into simplicial complex. However, such calculations are tedious. We will learn more tools to compute cohomology rings later.

First, note that the Klein bottle can be realized as the CW complex



The cellular structure is

$$C_*^{\text{cell}}(K, \mathbb{F}_2): \quad 0 \longrightarrow \underset{\deg 2}{\mathbb{F}_2\{u, v\}} \longrightarrow \underset{\deg 1}{\mathbb{F}_2\{a, b, c\}} \longrightarrow \underset{\deg 0}{\mathbb{F}_2\{p\}} \longrightarrow 0,$$

and the dual cellular chain complex is

$$C^*_{\text{cell}}(K, \mathbb{F}_2): \quad 0 \longrightarrow \underset{\deg 0}{\mathbb{F}_2\{\delta_p\}} \longrightarrow \underset{\deg -1}{\mathbb{F}_2\{\delta_a, \delta_b, \delta_c\}} \longrightarrow \underset{\deg -2}{\mathbb{F}_2\{\delta_u, \delta_v\}},$$

where δ_p is a function that evaluates to 1 at p and 0 at any of other cells. Other functions are defined similarly.

We compute a bunch of delta maps. For the coboundary of δ_p, we note that $\partial a = \partial b = \partial c = 0$, so we have that $\partial \delta_p = 0$. For the coboundary of δ_a, δ_b, δ_c, first, note that $\partial u = \partial v = a + b + c$ (because we are working in $\mathbb{F}_2$). Thus,

$$\begin{aligned} \partial \delta_a(u) &= \delta_a(\partial u) = \delta_a(a + b + c) = 1 \\ \partial \delta_a(v) &= \delta_a(\partial v) = \delta_a(a + b + c) = 1 \\ \implies \partial \delta_a &= \delta_u + \delta_v \end{aligned}$$

Similarly,

$$\partial \delta_b(u) = \partial \delta_b(v) = \partial \delta_c(u) = \partial \delta_c(v) = 1 \implies \partial \delta_b = \partial \delta_c = \delta_u + \delta_v.$$

Next, we read off the cohomology group as

$$H^i(K, \mathbb{F}_2) = \begin{cases} \mathbb{F}_2\{\overline{\delta_p}\} & i = 0 \\ \mathbb{F}_2\{\overline{\delta_a + \delta_b}, \overline{\delta_a + \delta_c}\} & i = 1 \\ \mathbb{F}_2\{\overline{\delta_u}\} & i = 2. \end{cases}$$

We let

$$1 = \overline{\delta_p}, \quad \alpha = \overline{\delta_a + \delta_b}, \quad \beta = \overline{\delta_b + \delta_c}, \quad K = \overline{\delta_u},$$

so we have $H^i(K, \mathbb{F}_2) \simeq \mathbb{F}_2\{1, \alpha, \beta, K\}$ as modules. We clearly have $K^2 = K\alpha = K\beta = 0$ since these have too large degree.

We now compute the more interesting product of α^2, β^2, and $\alpha\beta$. First, we compute α^2. Note that

$$\begin{aligned} (\alpha^2)(u) &= \Big((\delta_a + \delta_b)(\text{front 1-face of } u)\Big)\Big((\delta_a + \delta_b)(\text{back 1-face of } u)\Big) \\ &= \Big((\delta_a + \delta_b)(a)\Big)\Big((\delta_a + \delta_b)(b)\Big) \\ &= 1 \cdot 1 = 1 \\ (\alpha^2)(v) &= \Big((\delta_a + \delta_b)(\text{front 1-face of } v)\Big)\Big((\delta_a + \delta_b)(\text{back 1-face of } v)\Big) \\ &= \Big((\delta_a + \delta_b)(c)\Big)\Big((\delta_a + \delta_b)(a)\Big) \\ &= 0 \cdot 1 = 0, \end{aligned}$$

which implies that $\alpha^2 = \delta_u$, or $\alpha^2 = K$.

Next, we compute β^2.

$$\begin{aligned} (\beta^2)(u) &= \Big((\delta_b + \delta_c)(\text{front 1-face of } u)\Big)\Big((\delta_a + \delta_c)(\text{back 1-face of } u)\Big) \\ &= \Big((\delta_b + \delta_c)(a)\Big)\Big((\delta_b + \delta_c)(b)\Big) \\ &= 0 \cdot 1 = 0 \\ (\beta^2)(v) &= \Big((\delta_b + \delta_c)(\text{front 1-face of } v)\Big)\Big((\delta_b + \delta_c)(\text{back 1-face of } v)\Big) \\ &= \Big((\delta_b + \delta_c)(c)\Big)\Big((\delta_b + \delta_c)(a)\Big) \\ &= 1 \cdot 0 = 0, \end{aligned}$$

which implies that $\beta^2 = 0$. Finally,

$$\begin{aligned} (\alpha\beta)(u) &= \Big((\delta_a + \delta_b)(\text{front 1-face of } u)\Big)\Big((\delta_b + \delta_c)(\text{back 1-face of } u)\Big) \\ &= \Big((\delta_a + \delta_b)(a)\Big)\Big((\delta_b + \delta_c)(b)\Big) \\ &= 1 \cdot 1 = 1 \\ (\alpha\beta)(v) &= \Big((\delta_a + \delta_b)(\text{front 1-face of } v)\Big)\Big((\delta_b + \delta_c)(\text{back 1-face of } v)\Big) \\ &= \Big((\delta_a + \delta_b)(c)\Big)\Big((\delta_b + \delta_c)(a)\Big) \\ &= 0 \cdot 0 = 0, \end{aligned}$$

which implies that $\alpha\beta = K$.

Finally, the ring structure is

$$\boxed{H^*(K, \mathbb{F}_2) \simeq \frac{\mathbb{F}_2[1, \alpha, \beta]}{\langle \alpha^2 = \alpha\beta = K,\ \alpha^4 = 0,\ \beta^2 = 0 \rangle}},$$

where $\deg\alpha = \deg\beta = 1$.

§7.3.3 Properties of Cohomology Rings

As seen from the above example, computation of cohomology rings can be done through a routine process starting from semisimplicial set. However, this process is long and tedious. We will study some properties that allow us to do this more quickly.

Our goal is to compute the cohomology of the torus Σ and the mickey mouse space M, as in Example 7.0.1, and show that these are different.

▶ **Cohomology ring is a functor.** The constructions that we have is functorial. Thus, for any continuous map $f : X \to Y$, we have a morphism of graded ring

$$H^*(f) : H^*(Y) \to H^*(X),$$

which is useful in both computing cohomology rings and using cohomology rings to deduce properties of maps between topological spaces.

▶ **Cohomology ring of a product.** We now explain how to compute cohomology of the product. Recall the cross product gives a map

$$\times : H^*(X, R) \otimes_R H^*(Y, R) \to H^*(X \times Y, R).$$

It is not guaranteed that this is not an isomorphism. However, it often is. For example, it turns out that $\times$ is an isomorphism if R is a PID and $H_*(X, R)$ is a free R-module, which cover most of our use cases.

We now give the cup product structure of $H^*(X, R) \otimes_R H^*(Y, R)$. Since the theorem is mouthful to state, we omit the coefficient ring R.

Lemma 7.3.5.

The following diagram commutes.

$$\begin{tikzcd}
(H^*(X) \otimes_R H^*(Y)) \otimes_R (H^*(X) \otimes_R H^*(Y)) \arrow[r, "\smile"] \arrow[d, "\times"] & H^*(X) \otimes_R H^*(Y) \arrow[d, "\times"] \\
H^*(X \times Y) \otimes_R H^*(X \times Y) \arrow[r, "\smile"] & H^*(X \times Y),
\end{tikzcd}$$

where the cup product on $H^*(X) \otimes_R H^*(Y)$ is defined by

$$(a_1 \otimes b_1) \smile (a_2 \otimes b_2) = (-1)^{\deg a_2 \deg b_1} \Big((a_1 \smile a_2) \otimes (b_1 \smile b_2) \Big).$$

▶ *Proof.* This is more or less checking definition, utilizing the fact that cross product is associative (proven in the course of Theorem 7.3.4). Let $\Delta_X : X \to X \times X$, $\Delta_Y : Y \to Y \times Y$, and $\Delta_{X \times Y} : X \times Y \to (X \times Y) \times (X \times Y)$ be the diagonal maps. Note that $\Delta_{X \times Y}$ factors as

$$\begin{tikzcd}
X \times Y \arrow[rr, "\Delta_{X \times Y}"] \arrow[dr, "\Delta_X \times \Delta_Y"'] & & X \times Y \times X \times Y \\
& X \times X \times Y \times Y \arrow[ur, "1 \times T \times 1"'] &
\end{tikzcd}$$

where $T : X \times Y \to Y \times X$ is the map obtained by swapping coordinates. Note that from the diagram (7.2), we get that $T^*(a \times b) = (-1)^{\deg a \deg b}(b \times a)$.

We now compute

$$\begin{aligned}
(a_1 \times b_1) \smile (a_2 \times b_2) &= (\Delta_{X \times Y})^* \, (a_1 \times b_1 \times a_2 \times b_2) \\
&= (\Delta_X \times \Delta_Y)^* \, (1 \times T \times 1)^* \, (a_1 \times b_1 \times a_2 \times b_2) \\
&= (\Delta_X \times \Delta_Y)^* \, (a_1 \times T^*(b_1 \times a_2) \times b_2) \\
&= (-1)^{\deg b_1 \deg a_2} (\Delta_X \times \Delta_Y)^* \, (a_1 \times a_2 \times b_1 \times b_2).
\end{aligned}$$

By naturality of cross product, the following diagram commutes.

$$\begin{tikzcd}
H^*(X \times X) \otimes_R H^*(Y \times Y) \arrow[r, "\times"] \arrow[d, "\Delta_X^* \otimes \Delta_Y^*"] & H^*(X \times X \times Y \times Y) \arrow[d, "(\Delta_X \times \Delta_Y)^*"] \\
H^*(X) \otimes_R H^*(Y) \arrow[r, "\times"] & H^*(X \times Y)
\end{tikzcd}$$

Starting from $(a_1 \times a_2) \otimes_R (b_1 \times b_2)$ and tracing through the two path gives

$$\begin{aligned}(\Delta_X \times \Delta_Y)^* (a_1 \times a_2 \times b_1 \times b_2) &= \Delta_X^*(a_1 \times a_2) \ \times \ \Delta_Y^*(b_1 \times b_2) \\ &= (a_1 \smile a_2) \times (b_1 \smile b_2),\end{aligned}$$

which gives

$$(a_1 \times b_1) \smile (a_2 \times b_2) = (-1)^{\deg b_1 \deg a_2}(a_1 \smile a_2) \times (b_1 \smile b_2),$$

showing that the diagram commutes. □

This allows us to compute cohomology ring of a product easily, as demonstrated by the following example.

Example 7.3.6 (Cohomology ring of a torus).

Recall that

$$H^i(S^1, \mathbb{Z}) = \begin{cases} 0 & i \neq 0, 1 \\ \mathbb{Z}\{1\} & i = 0 \\ \mathbb{Z}\{\alpha\} & i = 1, \end{cases}$$

and $\alpha^2 = 0$. By taking cross product and using Proposition 7.3.2, we get that

$$H^i(\Sigma, \mathbb{Z}) = \begin{cases} \mathbb{Z}\{1 \times 1\} & i = 0 \\ \mathbb{Z}\{\alpha \times 1, 1 \times \alpha\} & i = 1 \\ \mathbb{Z}\{\alpha \times \alpha\} & i = 2 \\ 0 & \text{otherwise.} \end{cases}$$

We compute

$$(\alpha \times 1)(1 \times \alpha) = \alpha \times \alpha, \quad (1 \times \alpha)(\alpha \times 1) = -(\alpha \times \alpha)$$

(Note the minus sign; very annoying.) Other products are zero. Thus, if $a = \alpha \times 1$ and $b = 1 \times \alpha$, then

$$H^*(\Sigma, \mathbb{Z}) = \boxed{\frac{\mathbb{Z}[a, b]}{\langle ab = -ba, \quad a^2 = b^2 = 0\rangle}}.$$

▶ **Cohomology of disjoint union.** Unsurprisingly, we have the following proposition:

Proposition 7.3.7.

Let X and Y be topological spaces. Then,

$$H^*(X \amalg Y, R) \simeq H^*(X, R) \times H^*(Y, R),$$

where we are taking ring product. Term (a, b) where $a \in H^*(X, R)$ and $b \in H^*(Y, R)$ is said to have degree d if and only if $\deg a = \deg b = d$.

▶ *Proof.* The inclusion $X \to X \amalg Y$ and $Y \to X \amalg Y$ induces maps

$$H^*(X \amalg Y, R) \to H^*(X, R), \quad H^*(X \amalg Y, R) \to H^*(Y, R),$$

which then induce the map of cohomology rings

$$H^*(X \amalg Y, R) \to H^*(X, R) \times H^*(Y, R).$$

It is easy to see that this is an isomorphism at the cohomology level. Since we already know it preserves the ring structure, it follows that this is an isomorphism of cohomology ring. $\square$

Example 7.3.8 (Cohomology of mickey mouse space.)**.**

We now compute the cohomology of mickey mouse space $M = S^1 \vee S^2 \vee S^2$ over $\mathbb{Z}$. Note that M is a quotient of $S^2 \amalg S^1 \amalg S^1$ identifying one point from each of the spheres to a point, which gives a surjective map

$$S^2 \amalg S^1 \amalg S^1 \to M,$$

which then induces the map

$$H^*(M, \mathbb{Z}) \to H^*(S^2 \amalg S^1 \amalg S^1, \mathbb{Z}).$$

We can compute the cohomology groups of M and of $S^2 \amalg S^1 \amalg S^1$ in many ways (e.g., Mayer-Vietoris). We will find that the above map induces isomorphism in H^1 and H^2. Thus, if $H^1(M, \mathbb{Z}) = \mathbb{Z}a \oplus \mathbb{Z}b$, then from the cup product structure of $S^2 \amalg S^1 \amalg S^1$, we get that $ab = 0$, since a corresponds to $(0, \omega, 0) \in H^*(S^2 \sqcup S^1 \sqcup S^1, \mathbb{Z})$ and b corresponds to $(0, 0, \omega) \in H^*(S^2 \sqcup S^1 \sqcup S^1, \mathbb{Z})$, where ω is a generator of $H^*(S^1, \mathbb{Z})$. Thus, their product is 0, or $\boxed{ab = 0}$.

We have another generator $H^2(M, \mathbb{Z}) = \mathbb{Z}x$, and so the cohomology ring is

$$H^*(M, \mathbb{Z}) \simeq \mathbb{Z}a \oplus \mathbb{Z}b \oplus \mathbb{Z}x$$

where $\deg a = \deg b = 1$ and $\deg x = 2$ and any pair of cup product vanishes.

Finally, we can now show that M and Σ are not homotopy equivalent! The two degree 1 generators of M multiply to 0, but this is not the case for Σ.

§7.4 Problems

Problem 7.A. Compute $H^q(\mathbb{RP}^n, \mathbb{Z})$ for all q and n.

Problem 7.B. Note that $\mathbb{R}^n$ is the universal cover of $(S^1)^n \simeq \mathbb{R}^n/\mathbb{Z}^n$. Let M be an $n \times n$ matrix with entries in $\mathbb{Z}$, which defines a linear transformation $\mathbb{R}^n \to \mathbb{R}^n$. This map descends to a map $(S^1)^n \to (S^1)^n$. Describe the effect of this map on $H_1((S^1)^n, \mathbb{Z})$ and on $H_n((S^1)^n, \mathbb{Z})$.

Problem 7.C. Compute the cohomology ring $H^*(\Sigma_g, \mathbb{Z})$ of Σ_g, the g-holed torus.

(Hint: consider the quotient map $\Sigma_g \to \underbrace{\Sigma_1 \vee \cdots \vee \Sigma_1}_{g}$.)

Problem 7.D. (a) Suppose that there exists a map $f : \Sigma_g \to \Sigma_h$ such that $H_2(f)$ is nonzero. Prove that $g \geq h$.

(b) Conversely, prove that if $g \geq h$, there is a map $f : \Sigma_g \to \Sigma_h$ that induces an isomorphism on H_2.

8 Poincaré Duality

In this chapter, we specialize to manifolds and work towards an important homological property of manifolds called Poincaré Duality. This has a number of applications, including computation of cohomology rings of some spaces, and Lefschetz fixed point theorem.

First, let us quickly recall what manifolds are.

Definition 8.0.1.

A Hausdorff topological space X is an **n-manifold** if every point $x \in X$ has an open neighborhood homeomorphic to $\mathbb{R}^n$.

Remark 8.0.2.

Some authors require a manifold to be second-countable, i.e., have a countable basis of open sets.

Example 8.0.3.

Here are some examples of manifolds.

- $\mathbb{R}^n$, S^n, and $\mathbb{RP}^n$ are n-manifolds.
- $\mathbb{CP}^n$ is a $2n$-manifold.
- A torus $S^1 \times S^1$ and a Klein bottle are 2-manifolds.

On the other hand, two spheres kissing at a point is not a 2-manifold the neighborhood around the kissing point does not look like an open set of $\mathbb{R}^2$.

§8.1 Poincaré Duality on $\mathbb{F}_2$

Since the statement of Poincaré duality is a bit complex, we give a simple version over $\mathbb{F}_2$ first and explore some consequences.

§8.1.1 Statement over $\mathbb{F}_2$

Theorem 8.1.1 (Poincaré Duality I).

Suppose M is a compact, connected n-manifold. Then

(a) We have $H^n(M, \mathbb{F}_2) \simeq \mathbb{F}_2$.

(b) The cup product

$$H^p(M, \mathbb{F}_2) \times H^q(M, \mathbb{F}_2) \xrightarrow{\smile} H^n(M, \mathbb{F}_2) \simeq \mathbb{F}_2$$

is a perfect pairing.

This is not the most general version of Poincaré Duality. In order to prove it, we will state two more versions, Theorem 8.3.3 (for arbitrary ring) and Theorem 8.3.9 (for non-compact manifolds), with increasing generality. Then, we will prove the most general version there.

Using universal coefficient theorem, we get the following consequence.

Corollary 8.1.2.

If M is a compact n-manifold, then for all $p+q=n$, we have natural isomorphisms between

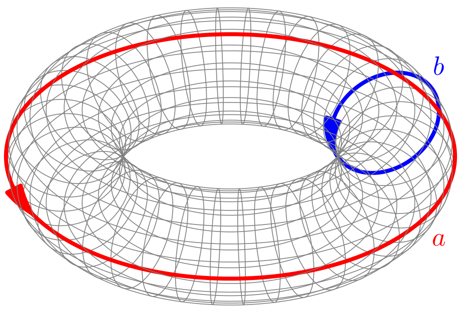
$$\operatorname{Hom}(H_p(M, \mathbb{F}_2), \mathbb{F}_2) \simeq H^p(M, \mathbb{F}_2) \simeq \operatorname{Hom}(H^q(M, \mathbb{F}_2), \mathbb{F}_2) \simeq H_q(M, \mathbb{F}_2).$$

► **Geometric intuition behind Poincaré Duality** The theorem tells you about the *symmetry* between p-th homology and q-th homology. It gives you the correspondence between

$$\{q\text{-dimensional objects}\} \longleftrightarrow \{\text{function on } p\text{-dimensional objects}\},$$

and roughly speaking, the correspondence is given by for each q-dimensional object x, we get a function that takes a p-dimensional object y and output the number of intersection points between x and y. For dimension reason, we can perturb x and y slightly so that generically, the intersection of x and y are a bunch of points, and we can count them.

As an example, if $M = S^1 \times S^1$, then $H_1(M, \mathbb{F}_2)$ is generated by two loops $[a]$ and $[b]$, where $[a]$ and $[b]$ wrap in a different direction as shown below.



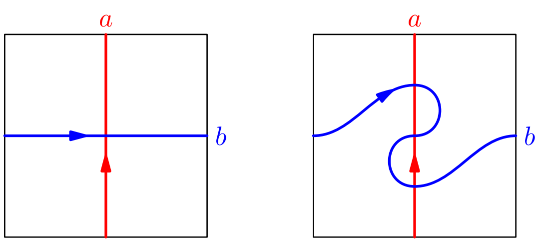
Thus, Poincaré duality tells us that there are functions $f_{[a]}$ and $f_{[b]}$ in $H^1(M, \mathbb{F}_2)$.

What are these functions? It turns out that

$$f_{[a]}([a]) = 0, \quad f_{[a]}([b]) = 1,$$

corresponding to the fact that $[a]$ and $[b]$ intersect at 1 point.

Notice that this does not depend on the choice of deformation of a and b. For example, supoose that one starts with a and b intersecting at one point like the left figure below. In that case $f_{[b]}([a]) = 1$. If one deform b slightly, making it look like the right figure, then there are 3 intersections. However, $3 = 1$ in $\mathbb{F}_2$.



If we want to make this work for general ring, e.g., $\mathbb{Z}$, then we need to assign the orientation of the intersection. Intuitvely, the middle intersection has opposite orientation to all other, so the intersection number between $[a]$ and $[b]$ is 1. We will discuss orientation in the next section.

One caveat is that loops should be in **generic position**. You should not cheat by for example, making $[b]$ tangent to $[a]$, causing 2 intersections. Formalizing what a generic position mean is difficult, but algebraic topology automatically capture this geometric structure without even defining what a generic position mean.

§8.1.2 Example Applications

Let us now demonstrate applications of Poincaré duality. The most immediate application is to determine the dimension of H_p from the dimension of H_q. For example, if M is a compact 3-manifold, and I know that $H_0(M, \mathbb{F}_2) \simeq \mathbb{F}_2 \oplus \mathbb{F}_2$, I immediately know that $H_3(M, \mathbb{F}_2) \simeq \mathbb{F}_2 \oplus \mathbb{F}_2$. Similarly, if I know that $H_1(M, \mathbb{F}_2) \simeq \mathbb{F}_2 \oplus \mathbb{F}_2 \oplus \mathbb{F}_2$, I immediately know that $H_2(M, \mathbb{F}_2) \simeq \mathbb{F}_2 \oplus \mathbb{F}_2 \oplus \mathbb{F}_2$.

In the case of torus, one can see that this is true because the homology groups of the torus are $\mathbb{F}_2$ for $i = 0$, $\mathbb{F}_2 \oplus \mathbb{F}_2$ for $i = 1$, and $\mathbb{F}_2$ for $i = 2$.

However, this is not the only power that Poincaré duality offers. Exploiting the perfect pairing structure allows you to compute the cohomology ring quickly. The most important consequence is

Corollary 8.1.3.

Let M be a compact and connected n-manifold. If $x \in H^p(M, \mathbb{F}_2)$, then there exists $y \in H^q(M, \mathbb{F}_2)$ such that $xy \neq 0$.

▶ *Proof.* If there exists x such that $xy = 0$ for all y, then the pairing in Theorem 8.1.1 is not perfect, a contradiction. □

▶ **Cohomology ring of real projective spaces** We compute $H^*(\mathbb{RP}^n, \mathbb{F}_2)$.

Example 8.1.4 (Cohomology ring of $\mathbb{RP}^2$).

Consider $\mathbb{RP}^2$. The cohomology group is given by

$$H^i(\mathbb{RP}^2, \mathbb{F}_2) = \begin{cases} \mathbb{F}_2\{1\} & i = 0 \\ \mathbb{F}_2\{x\} & i = 1 \\ \mathbb{F}_2\{y\} & i = 2 \\ 0 & i \neq 0, 1, 2. \end{cases}$$

To compute the cohomology ring, the only non-obvious product is x^2, which could be either 0 or y. However, Poincaré duality tells us immediately that $x^2 = y$. Why? Note that in the composition

$$\begin{array}{ccccc} H^1(\mathbb{RP}^2, \mathbb{F}_2) \otimes_{\mathbb{F}_2} H^1(\mathbb{RP}^2, \mathbb{F}_2) & \longrightarrow & H^2(\mathbb{RP}^2, \mathbb{F}_2) & \longrightarrow & \mathbb{F}_2 \\ x \otimes x & \longrightarrow & x^2 & & \end{array}$$

However, if $x \otimes x = 0$, then the pairing would be degenerate at x. Hence, the only possibility is $x^2 = y$. Consequently, we have the cohomology ring

$$H^*(\mathbb{RP}^2, \mathbb{F}_2) \simeq \frac{\mathbb{F}_2[x]}{(x^3)}.$$

Let us do another example.

Example 8.1.5 (Cohomology ring of $\mathbb{RP}^3$).

The cohomology group is given by

$$H^i(\mathbb{RP}^3, \mathbb{F}_2) = \begin{cases} \mathbb{F}_2\{1\} & i = 0 \\ \mathbb{F}_2\{x\} & i = 1 \\ \mathbb{F}_2\{y\} & i = 2 \\ \mathbb{F}_2\{z\} & i = 3. \end{cases}$$

The trivial products are $xz = y^2 = yz = z^2 = 0$. The nontrivial products are x^2 and xy.

- To compute x^2, note that the map from $\mathbb{RP}^2 \to \mathbb{RP}^3$ induces a map on cohomology rings. Thus, we have $x^2 = y$.
- On the other hand, Poincaré duality gives that $xy = z$.

Hence, we deduce that

$$H^*(\mathbb{RP}^3, \mathbb{F}_2) \simeq \frac{\mathbb{F}_2[x]}{(x^4)}.$$

By repeating this argument and using induction, it's not hard to show that

$$\boxed{H^*(\mathbb{RP}^n, \mathbb{F}_2) \simeq \frac{\mathbb{F}_2[x]}{(x^{n+1})}}. \tag{8.1}$$

► **Euler Characteristic of Odd-dimensional Manifold.** Finally, here is an easy fact about Euler characteristic of odd-dimensional manifold.

Corollary 8.1.6.

Let M be an odd-dimensional compact manifold. Then, $\chi(M) = 0$.

► *Proof.* By Problem 8.B, $\dim H_i(M, \mathbb{F}_2)$ is finite. By definition of Euler characteristic, we have

$$\chi(M) = \sum_{i=0}^{n} (-1)^i \dim H_i(M, \mathbb{F}_2).$$

However, by Poincaré duality, we have $\dim H_i(M, \mathbb{F}_2) = \dim H_{n-i}(M, \mathbb{F}_2)$, so

$$\begin{aligned} \chi(M) = &= \sum_{i=0}^{n} (-1)^i \dim H_{n-i}(M, \mathbb{F}_2) \\ &= \sum_{i=0}^{n} -(-1)^i \dim H_i(M, \mathbb{F}_2). \end{aligned}$$

Adding those equations gives $2\chi(M) = 0$, or $\chi(M) = 0$. □

§8.2 Orientation of a Manifold

As discussed in the above section, we expect Poincaré Duality to work for only orientable manifolds. We now define this rigorously.

§8.2.1 Local and Global Orientation

Given any ring R, we will equip a manifold M with a global R-orientation. Given such orientation, we will then get Poincaré duality with coefficients in R. All manifold have unique global $\mathbb{F}_2$-orientation, which is why the statement of Poincaré duality is simple in the case of $\mathbb{F}_2$.

We begin by considering the orientation locally.

Definition 8.2.1 (Local Homology)**.**

Suppose M is an n-manifold, $x \in M$, and R is a ring. Then, the **local homology** of M with coefficients in R is

$$H_n(M, M - \{x\}, R)$$

(the equal sign is just recalling definition of relative homology).

We now compute what this is. Excision theorem (Theorem 4.4.1) say that this is the same as

$$H_n(U, U - \{x\}, R)$$

for any open neighborhood U of x. We now take U that is homeomorphic to $\mathbb{R}^n$, so we have that

$$H_n(M, M - \{x\}, R) \simeq H_n(\mathbb{R}^n, \mathbb{R}^n - \{x\}, R).$$

Snake lemma (Theorem 4.3.11) gives the long exact sequence

$$\begin{tikzcd} H_n(\mathbb{R}^n - \{x\}, R) \arrow[r] & H_n(\mathbb{R}^n, R) \arrow[r] & H_n(\mathbb{R}^n, \mathbb{R}^n - \{x\}, R) \arrow[dll, "\partial"'] \\ H_{n-1}(\mathbb{R}^n - \{x\}, R) \arrow[r] & H_{n-1}(\mathbb{R}^n, R) \arrow[r] & \dots \end{tikzcd},$$

and filling in this diagram (noting that $\mathbb{R}^n - \{x\}$ is homotopic to S^{n-1}) gives

$$\begin{tikzcd} 0 \arrow[r] & 0 \arrow[r] & H_n(\mathbb{R}^n, \mathbb{R}^n - \{x\}, R) \arrow[dll, "\partial"'] \\ R \arrow[r] & 0 \arrow[r] & \dots \end{tikzcd},$$

which allows us to conclude that

$$H_n(M, M - \{x\}, R) \;\simeq\; H_n(\mathbb{R}^n, \mathbb{R}^n - \{x\}, R) \;\simeq\; R.$$

Definition 8.2.2.

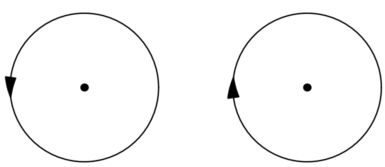
A **local R-orientation** of M at x is an R-module generator of

$$H_n(M, M - \{x\}, R).$$

Notice that this is up to multiplication by unit $R^\times$.

Notice that local orientation depends on multiplication by a unit. In particular,

- With coefficient ring $\mathbb{Z}$, there are 2 local orientations at every point. If M is 2-dimensional manifold, then the two local orientations correspond to clockwise and counterclockwise:



This geometric intuition generalizes for n-dimensional manifolds: there are still two orientations.

- With coefficient ring $\mathbb{F}_2$, there is a unique local orientation.

Unfortunately, the isomorphism from $H_n(M, M-\{x\}, R)$ to R is not canonical (i.e., there is no consistent choice of unit). To go from local to global orientation, we have to pick units in a consistent way.

Definition 8.2.3.

For an n-manifold M, define the set

$$\begin{aligned} o_M \otimes R &= \coprod_{x \in M} H_n(M, M-\{x\}, R) \\ &= \{(x, \alpha) : \alpha \in H_n(M, M-\{x\}, R)\}. \end{aligned}$$

(The coproduct is as sets, hence the description in the second line.) This is a topological space, and the topology is in the text following this definition.

(Remark: we often write $o_M = o_M \otimes \mathbb{Z}$. Also, a warning that this is not a tensor product).

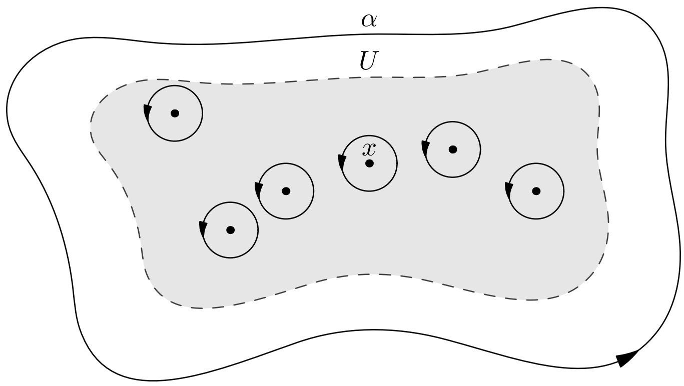
We define topology on $o_M \otimes R$. For any closed set $A \subset M$, define the natural map

$$j_{A,x} : H_n(M, M-A, R) \;\to\; H_n(M, M-\{x\}, R)$$

from an inclusion $M - A \to M \setminus \{x\}$. For any open set $U \subset M$ and $\alpha \in H_n(M, M-\overline{U}, R)$, define

$$V_{U,\alpha} = \{(x, j_{\overline{U},x}(\alpha)) : x \in U\} \quad \subset o_M \otimes R,$$

which will be an open set of $o_M \otimes R$. In more plain language, the set of local orientation arises from the same $H_n(M, M-\overline{U}, R)$ should be considered "together", hence open. For example, when $R = \mathbb{Z}$, then the set $V_{U,\alpha}$ consists of the set of pairs (x, ε) for which $x \in U$ and ε all have the same orientation. Some elements of $V_{U,\alpha}$ are shown below.



The set

$$\{V_{U,\alpha} : U \text{ open}, \ \alpha \in H_n(M, M-\overline{U}, R)\}$$

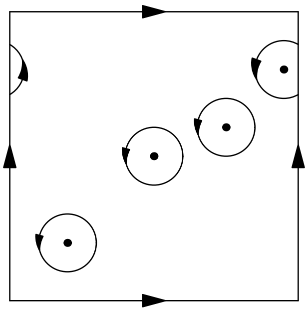
forms a basis for the topology of $o_M \otimes R$. Then the **projection map** $p : o_M \otimes R \to M$ (sending (x, α) where $\alpha \in H_n(M, M-\{x\}, R)$ to x) is a covering map. (We leave the verification of this fact as an exercise to the reader.) In other words, $o_M \otimes R$ is a covering space of M.

Definition 8.2.4.

A **section** of p is a continuous map $f : M \to o_M \otimes R$ such that $p \circ f = \mathrm{id}_M$. (In other words, for each $x \in M$, we assign $(x, \alpha) \in o_M \otimes R$.)

A **global R-orientation** of M is a continuous section $f : M \to o_M \otimes R$ of p with additional condition that for each $x \in M$, $f(x) = (x, \alpha)$ for some **unit** $\alpha \in H_n(M, M - \{x\}, R)^\times$.

The continuity condition guarantees that the orientation is consistent along every open set $U \subseteq X$. For example, a torus has a global orientation. Viewed as a quotient of $[0,1]^2$, the global orientation is simply taking point x and then assign a consistent orientation for the whole torus.



We now investigate a condition to tell whether a global R-orientation of a manifold exists or not. In order to gain a complete picture, we study this in two perspectives.

- **Orientation Theorem**, which charaterizes all sections of R in terms of the global homology group $H_n(M, R)$. This will be done in Section 8.2.2
- **Covering Spaces Theory**, which charaterizes all sections of R in terms of the fundamental group $\pi_1(M)$. This will be done in Section 8.2.3.

Together, these two perspectives will give a nice condition for a manifold to be R-oriented.

§8.2.2 Orientation Theorem

Theorem 8.2.5 (Orientation Theorem)**.**

If M is a connected, compact manifold, then there is a bijection between

$$\{\text{sections of } p\} \longleftrightarrow H_n(M, R).$$

Under this bijection, we declare that a global orientation of M corresponds to the **fundamental class** $[M] \in H_n(M, R)$.

► *Proof Idea.* We only describe the bijection because in order to show that this is a bijection, we need to formulate a general statement. For every $x \in M$, there is a map

$$\phi_x : H_n(M, R) \to H_n(M, M - \{x\}, R).$$

Assembling this for each x, we get a map

$$\begin{aligned} H_n(M, R) &\to \{\text{sections of } p\} \\ [\alpha] &\mapsto (x \mapsto (x, \phi_x(\alpha))), \end{aligned}$$

which is clearly a section. It is not so hard to check that this is continuous. □

Unfortunately, proving that the above map is an isomorphism requires formulating a general statement about compact subset. This will introduce you to an idea that will soon be recurring: to prove a statement about compact manifolds, we make a general statement about compact **subsets** of manifolds, and we prove it by induction, slowly growing our manifold.

In the case of Theorem 8.2.5, to state the generalization, we make the following definition.

Definition 8.2.6.

For any compact subset A of a manifold M, define

$$\Gamma(A, o_M \otimes R) = \left\{ \begin{array}{c} \text{continuous map } A \to \coprod_{x \in A} H_n(M, M - \{x\}, R) \ \subset o_M \otimes R \\ \text{that is a continuous section of the projection map} \end{array} \right\}.$$

Then, similar to the above construction, there is a map

$$j_A : H_n(M, M - A, R) \to \Gamma(A, o_M \otimes R),$$

and the main claim is the following, which immediately implies Theorem 8.2.5.

Theorem 8.2.7.

The map j_A above is an isomorphism.

▶ *Proof.* The proof is in five steps.

▶ **Base case.** Suppose that A is a convex compact subset of $\mathbb{R}^n$. Then, due to homotopy equivalence,

$$\phi_x : H_n(\mathbb{R}^n, \mathbb{R}^n - A, R) \longrightarrow H_n(\mathbb{R}^n, \mathbb{R}^n - \{x\}, R)$$

is an isomorphism for all points $x \in A$. Assembling this map for all $x \in A$ gives

$$\begin{aligned} H_n(\mathbb{R}^n, \mathbb{R}^n - A, R) &\to \Gamma(A, o_M \otimes R) \\ \alpha &\mapsto (x \mapsto \phi_x(\alpha)), \end{aligned}$$

which we can check that it is an isomorphism due to that ϕ_x is an isomorphism.

▶ **Proof for Union.** Suppose that we know the statement for A_1, A_2, and $A_1 \cap A_2$. Then, we prove it for $A_1 \cup A_2$. To prove this, we apply Mayor Vietoris to get the exact sequence (note that the first two terms are H_{n+1}, which are zero):

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & H_n(M, M - (A_1 \cup A_2)) \arrow[r] \arrow[d] & H_n(M, M - A_1) \oplus H_n(M, M - A_2) \arrow[r] \arrow[d] & H_n(M, M - (A_1 \cap A_2)) \arrow[d] \\ 0 \arrow[r] & 0 \arrow[r] & \Gamma(A_1 \cup A_2, o_M \otimes R) \arrow[r] & \Gamma(A_1, o_M \otimes R) \oplus \Gamma(A_2, o_M \otimes R) \arrow[r] & \Gamma(A_1 \cap A_2, o_M \otimes R). \end{tikzcd}$$

The first, second, fourth, and fifth down arrow are isomorphisms, so by five lemma, the third down arrow is an isomorphism.

▶ **Proof for finite union of convex compact sets.** Suppose that $A_1, \dots, A_n$ are n convex compact subsets of $\mathbb{R}^n$, we show that the theorem is true for $A_1 \cup \cdots \cup A_n$ by induction on n. The base case $n = 1$ is clear.

For the inductive step, we note that the theorem is true for $A_1 \cup \cdots \cup A_{n-1}$ (by induction hypothesis), A_n, and $(A_1 \cup \cdots \cup A_{n-1}) \cap A_n = (A_1 \cap A_n) \cup \cdots \cup (A_{n-1} \cap A_n)$ (by induction hypothesis again), hence it is true for $A_1 \cup \cdots \cup A_n$.

▶ **Proof for chain of intersection.** Suppose that we know the statement for all sets in the chain $A_1 \supseteq A_2 \supseteq \dots$, then we show that the statement is true for $A = \bigcap_{i=1}^{\infty} A_i$. The proof requires basic properties of colimits.

First, we note since simplices are compact, each simplex must lie in $X - A_i$ for some i. Therefore, the map

$$\operatorname*{colim}_{i\geq 1} S_n(X - A_i, R) \to S_n(X - A, R)$$

is surjective, and hence must be an isomorphism. The exact sequence $0 \to S_n(X - A_i, R) \to S_n(X, R) \to S_n(X, X - A_i, R) \to 0$ induces an exact sequence on their colimits, which then implies that

$$S_n(X, X - A, R) = \operatorname*{colim}_{i\geq 1} S_n(X, X - A_i, R).$$

Since colimit commutes with homology, we find that

$$H_n(X, X - A, R) = \operatorname*{colim}_{i\geq 1} H_n(X, X - A_i, R) = \operatorname*{colim}_{i\geq 1} \Gamma(A_i, o_M \otimes R)$$

(the last inequality follows from the statement from A_i). Finally, we note that the map

$$A_i \to \bigcup_{x\in A_i} H_n(M, M - \{x\}, R)$$

arising from the same entry in $H_n(X, X - A, R)$ agrees on the intersection, hence glue together to form a map $A \to \bigsqcup_{x\in A} H_n(M, M - \{x\}, R)$. One can check that this forms an isomorphism between

$$\operatorname*{colim}_{i\geq 1} \Gamma(A_i, o_M \otimes R) \to \Gamma(A, o_M \otimes R),$$

so we are done.

▶ **Why does this suffices?** First, we know that the theorem is true when $M = \mathbb{R}^n$ and A is a finite union of convex compact subset. Next, we note (without proof) the following fact: any compact subset $A \subseteq \mathbb{R}^n$ can be written as $A = \bigcap A_i$, where $A_0 \supseteq A_1 \supseteq A_2 \supseteq \dots$ such that each A_i is a finite union of convex compact subsets. Using this, one can prove the result for any compact subset A.

Now, suppose that M is arbitrary. Every subset of a manifold is contained in an infinite union of open neighborhoods of $\mathbb{R}^n$. However, every compact subset is contained in finitely many copies of $\mathbb{R}^n$, so every compact subset of M is a finite union of compact subsets for which we know the result. Hence, we are done. □

§8.2.3 Covering Space Theory and Orientation

Using classification theorem of covering spaces, we can classify sections of a covering map as follows.

Proposition 8.2.8.

Let X be a nice topological space and $x_0 \in X$. Let $p : (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ be a covering space. Then,

$$\{\text{sections of } p\} \simeq \{\text{fixed point of } p^{-1}(x_0) \text{ under } \pi_1(X, x_0)\text{-action}\}.$$

▶ *Proof.* The continuous section s of p can be thought of as the morphism from trivial covering and p:

$$\begin{tikzcd} X \arrow[rr, "s"] \arrow[dr, "\mathrm{id}"'] & & \widetilde{X} \arrow[dl, "p"] \\ & X & \end{tikzcd}$$

Thus, by Theorem 3.2.15, it corresponds to the morphism of $\pi_1(X, x_0)$-sets

$$\{\pi_1(X, x_0)\text{-set with one element}\} \to p^{-1}(x_0).$$

Let $\widetilde{x}_0 \in p^{-1}(x_0)$ be the image of that one element. Then, $\widetilde{x}_0$ is the fixed point. □

Applying Proposition 8.2.8 to the setting $o_M \otimes R$ gives

Corollary 8.2.9.

Suppose that M is compact and path-connected n-manifold with $x \in M$. Then,

$$H_n(M, R) \simeq \Gamma(M, o_M \otimes R) = \{\text{fixed points of } p^{-1}(x) \text{ under } \pi_1(M, x)\text{-action}\}.$$

We now analyze the action of $\pi_1(M, x)$ on the fiber of $o_M \otimes R$.

First, assume $R = \mathbb{Z}$. For any $\alpha \in \pi_1(M, x)$, the action of α on $p^{-1}(x)$ corresponds to multiplication by an element in $\mathbb{Z}^\times = \{\pm 1\}$, giving the group homomorphism

$$\phi : \pi_1(M, x) \to \mathbb{Z}^\times = \{\pm 1\},$$

and the action of α is by multiplication by $\phi(\alpha)$.

- If this map is trivial, then every element in $p^{-1}(x)$ is a fixed point. In particular, $o_M \otimes \mathbb{Z}$ is the disjoint union $\coprod_{n \in \mathbb{Z}} M$. Therefore, by Corollary 8.2.9, the set of sections corresponding to $\mathbb{Z}$, and two of those sections have value corresponding to the unit. In particular,
  - M is $\mathbb{Z}$-orientable.
  - $H_n(M, \mathbb{Z}) = \mathbb{Z}$ (by the orientation theorem Theorem 8.2.7).
- If this map is nontrivial, then since $1 \neq -1$ in $\mathbb{Z}$, there are no elements fixed by our $\pi_1(M, x)$-action. Therefore, by Corollary 8.2.9, there is no section, and so no orientation. In particular, we conclude that
  - M is not $\mathbb{Z}$-orientable.
  - $H_n(M, \mathbb{Z}) = 0$ (by the orientation theorem Theorem 8.2.7).

Now, for general R, once again, for any $\alpha \in \pi_1(M, x)$, the action of α on $p^{-1}(x)$ corresponds to multiplication by an element in $R^\times$. Moreover, the resulting morphism $\pi_1(M, x) \to R^\times$ must factor through that of $\mathbb{Z}^\times$:

$$\begin{tikzcd} \pi_1(M,x) \arrow[rr] \arrow[dr] & & R^\times \\ & \mathbb{Z}^\times \arrow[ur] & \end{tikzcd}$$

In particular, by a similar logic as above, we deduce the following theorem.

Theorem 8.2.10 (Orientability Condition)**.**

Let M be a connected, compact, n-dimensional manifold. Then one of the following holds.

(a) The map $\pi_1(M, x) \to \mathbb{Z}^\times$ is trivial, which implies that M is R-orientable for any ring R and $H_n(M, R) = R$.

We say that M is **orientable** in this case.

(b) The map $\pi_1(M, x) \to \mathbb{Z}^\times$ is nontrivial, which implies that $H_n(M, R) = R[2]$ and M is R-orientable if and only if every element of R is a 2-torsion.

We say that M is **non-orientable** in this case.

In particular, every manifold is $\mathbb{F}_2$-orientable. Moreover, if it is $\mathbb{Z}$-orientable, then it is R-orientable for any ring R.

Definition 8.2.11.

For any compact, connected, orientable manifold M and a ring R, the **fundamental class** is a generator $[M] \in H_n(M, R)$.

Example 8.2.12.

- A Klein bottle K is not orientable because $H_2(K, \mathbb{Z}) = 0$, not $\mathbb{Z}$.
- Similarly, the real projective planes of even dimension $\mathbb{RP}^{2n}$ is not orientable because $H_{2n}(\mathbb{RP}^{2n}, \mathbb{Z}) = 0$.
- On the other hand, the real projective planes of odd dimension $\mathbb{RP}^{2n-1}$ is orientable because $H_{2n-1}(\mathbb{RP}^{2n-1}, \mathbb{Z}) = \mathbb{Z}$.

We also note the following result.

Corollary 8.2.13.

If $H_1(M) = 0$, then M is orientable.

▶ *Proof.* The map $\pi_1(M, x) \to \mathbb{Z}^\times$ factors through the abelianization of $\pi_1(M, x)$, which is $H_1(M)$ (by Hurewicz's theorem Problem 4.D). Thus, if $H_1(M) = 0$, then the map $\pi_1(M, x) \to \mathbb{Z}^\times$ is trivial. □

§8.3 Poincaré Duality over General Coefficients

§8.3.1 Cap Product

In order to state a general Poincaré duality, we need to introduce another product.

Definition 8.3.1 (Cap Product)**.**

For any ring R and integers $p \le q$, there is the **cap product**

$$\frown : H^p(X, R) \otimes_R H_q(X, R) \to H_{q-p}(X, R)$$

induced from the result of the composition

$$S^p(X, R) \otimes_R S_q(X, R) \overset{\mathrm{id} \otimes \mathrm{AW}}{\longrightarrow} S^p(X, R) \otimes_R (S_p(X, R) \otimes_R S_{q-p}(X, R)) \longrightarrow S_{q-p}(X)$$

(the second map is a simple evaluation map $S^p(X, R) \otimes_R S_p(X, R) \to R$).

Once again, we leave the interested reader to check that this is a chain map (and in particular, see that there is no sign issue).

In the case where $p = q$, we get the map

$$H^p(X, R) \otimes H_p(X, R) \to H_0(X, R) = R,$$

which is defined by a simple evaluation map. This is called the **Kronecker pairing**, and we denote it as $\langle \bullet, \bullet \rangle$.

To spell the definition out on a chain level, suppose that $f \in S^p(X, R)$ and $\sigma : \Delta^q \to X$. Then we have

$$\boxed{f \frown \sigma = f(\sigma|_{\Delta^p, \text{front}}) \cdot (\sigma|_{\Delta^{q-p}, \text{back}})}. \tag{8.2}$$

It is straightforward to verify the following properies. (Both hold rue under chain level; there is no need to use acyclic model or construct a chain homotopy.)

Proposition 8.3.2 (Properties of Cap and Cup Products).

(a) For any $a, b \in H^*(X, R)$ and $\sigma \in H_*(X, R)$, we have

$$(a \smile b) \frown \sigma = a \frown (b \frown \sigma).$$

(In other words, cap product gives a $H^*(X, R)$-module structure on $H_*(X, R)$.)

(b) (Projection formula) For any $f : X \to X$, $a \in H^*(M, R)$, and $\sigma \in H_*(M, R)$, then

$$f_*(f^*(a) \frown \sigma) = a \frown f_*(\sigma).$$

§8.3.2 Statement of Poincaré Duality

Theorem 8.3.3 (Poincaré Duality II).

If M is a compact, connected n-manifold with global R-orientation, then for every integer p, the map

$$\begin{aligned} H^p(M, R) &\to H_{n-p}(M, R) \\ a &\mapsto a \frown [M] \end{aligned}$$

is an isomorphism (where $[M]$ is the fundamental class, Definition 8.2.11).

We now relate this version of Poincaré Duality to cup product. The following corollary shows that Theorem 8.3.3 implies Theorem 8.1.1.

Corollary 8.3.4.

Let M be a compact, connected n-manifold with global R-orientation. Suppose that $H_*(M, R)$ is a finitely-generated and free R-module. Then for any $p + q = n$, the cup product

$$\frown : H^p(M, R) \otimes_R H^q(M, R) \to H^n(M, R) \simeq R$$

is a perfect pairing.

▶ *Proof.* By universal coefficient theorem (since $H_*(M, R)$ is finitely-generated and free), the Kronecker pairing induces an isomorphism $H_q(M, R) \simeq \mathrm{Hom}(H^q(M, R), R)$. Thus, combining with the map from Theorem 8.3.3, we get an isomorphism $H^p(M, R) \to \mathrm{Hom}_R(H^q(M, R), R)$ sending a to the map that takes b to

$$\langle a \frown [M], b\rangle = b \frown (a \frown [M]) = (b \smile a) \frown [M],$$

where we used Proposition 8.3.2 (a).

Thus, this isomorphism induces the map $H^p(M, R) \otimes_R H^q(M, R) \to R$ that coincides with the cup product. Thus, we show that the cup product induces isomorphism $H^p(M, R) \to \mathrm{Hom}_R(H^q(M, R), R)$. Similarly, cup product induces isomorphism $H^q(M, R) \to \mathrm{Hom}_R(H^p(M, R), R)$. Hence, cup product is indeed a perfect pairing. □

§8.3.3 Poincaré Duality for Non-compact Manifolds

From now on, we drop the coefficient rings R from the cohomology or complex (so $H^n(M)$ means $H^n(M, R)$, for example).

The proof of Poincaré duality follows the same idea of the proof of Orientation Theorem Theorem 8.2.5. However, this is less direct because of the following.

Example 8.3.5 (The obvious base case fails)**.**

Poincaré duality is false for $\mathbb{R}^n$. In particular, $H_n(\mathbb{R}^n, R) = 0$, but $H^0(\mathbb{R}^n, R) = R$.

Thus, we need to think more carefully about the *base case* of the statement we want to use induction on. In particular, we need to somehow incorporate the compactness condition at the start (because $\mathbb{R}^n$ violates compactness). In order to do that, we define cohomology with compact support.

Definition 8.3.6.

For any manifold M, we let

$$\mathrm{cpt}(M) = \{\text{partially ordered set of compact subset of } M\},$$

where the order is by inclusion.

Partially ordered set $\mathrm{cpt}(M)$ can be thought as a category. This category is filtered: given any two compact sets K and L, there exists a compact subset $K \cup L$ above both K and L.

Definition 8.3.7.

We define **cohomology with compact support** as a filtered colimit

$$H^i_{\mathrm{cpt}}(M) = \operatorname*{colim}_{K \in \mathrm{cpt}(M)} H^i(M, M - K).$$

After we define cohomology with compact support, we now want to define the cap product on H^*_{cpt}.

Definition 8.3.8 (Relative cap product)**.**

For $A \subseteq X$, define the **relative cup product** to be a map

$$\frown: H^p(M, A) \otimes_R H_q(M, A) \to H_{q-p}(M).$$

At the chain level, we have to define the map

$$S^p(M, A) \otimes_R \frac{S_q(M)}{S_q(A)} \to S_{q-p}(M).$$

We define this to be the restriction of the cup product map

$$S^p(M) \otimes_R S_q(M) \to S_{q-p}(M).$$

One might worry tha this might not well-defined since any two lifts of an element of $S_q(M)/S_q(A)$ to $S_q(M)$ might yield different results. However, we claim this is not a concern. More specifically, show that the cap product send an element $f \in S^p(M, A)$ and $\sigma \in S_p(A)$ to 0. Note that this element is sent to

$$f \cap \sigma = f(\sigma|_{\Delta^p,\mathrm{front}}) \cdot (\sigma|_{\Delta^{q-p},\mathrm{back}}),$$

which is 0 because σ (and hence $\sigma|_{\Delta^p,\mathrm{front}}$) sits entirely in A, and f is 0 for any complex entirely in A.

Now, if M is connected manifold with R-orientation and fundamental class $[M] \in H_n(M, R)$.

Then, for any compact subset $K \subseteq M$, we get the map

$$\begin{aligned} H^p(M, M-K) &\to H_{n-p}(M) \\ \alpha &\mapsto \alpha \frown [M] \end{aligned}$$

where the cap product is taken relative to K. Taking colimit of this map yields

$$H^p_{\text{cpt}}(M) \to H_{n-p}(M).$$

Theorem 8.3.9 (Poincaré Duality III).

Let M be a connected manifold with R-orientation. Then the map described in the above paragraph is an isomorphism between $H^p_{\text{cpt}}(M) \to H_{n-p}(M)$.

If M is a compact manifold, then $\text{cpt}(M)$ has a maximal element, M, and so $H^i_{\text{cpt}}(M) = H^i(M)$. Thus, this theorem is a more general statement of Poincaré Duality II (Theorem 8.3.3).

Example 8.3.10 (Poincaré Duality III for $M = \mathbb{R}^n$).

Now, we can see why this is true for $M = \mathbb{R}^n$. For any compact set $K \subseteq \mathbb{R}^n$, there exists m such that $K \subseteq \overline{B_m(0)}$ (a closed ball centered at 0 with radius m). Then,

$$\begin{aligned} \operatorname*{colim}_{K \in \text{cpt}(\mathbb{R}^n)} H^i(\mathbb{R}^n, \mathbb{R}^n - K) &= \operatorname*{colim}_{m>0} H^i(\mathbb{R}^n, \mathbb{R}^n - \overline{B_m(0)}) \\ &= \operatorname*{colim}_{m>0} H^i(S^n) \\ &= \begin{cases} R & i = 0, n \\ 0 & \text{otherwise} \end{cases}. \end{aligned}$$

Warning 8.3.11.

H^*_{cpt} is not homotopy invariant (as seen in example for $\mathbb{R}^n$ above). It is not even functorial: not all maps $X \to Y$ induces a map $H^*_{\text{cpt}}(Y) \to H^*_{\text{cpt}}(X)$.

§8.3.4 Proof of Poincaré Duality III

We now prove Theorem 8.3.9 using induction argument. The "base case" $\mathbb{R}^n$ is done in Example 8.3.10. To develop the inductive step, we need something similar to Mayer-Vietoris sequence on H^*_{cpt}.

From now on, we let

$$H^*(X \mid Y) = H^*(X, X - Y)$$

and similarly $S^*(X \mid Y) = S^*(X, X - Y)$.

► **Mayer-Vietoris with compact support.** We now state Mayer-Vietoris on H^*_{cpt}.

Lemma 8.3.12 (Mayer-Vietoris with Compact Support).

There exists a long exact sequence

$$\cdots \to H^i_{\text{cpt}}(U \cap V) \to H^i_{\text{cpt}}(U) \oplus H^i_{\text{cpt}}(V) \to H^i_{\text{cpt}}(U \cup V) \to H^{i+1}_{\text{cpt}}(U \cap V) \ldots.$$

► *Proof.* For any compact subsets $K \subseteq U$ and $L \subseteq V$, we have the short exact sequence of chain complex

$$0 \longrightarrow S^*(M \mid K \cap L) \longrightarrow S^*(M \mid K) \oplus S^*(M \mid L) \longrightarrow S^*(M \mid K + L) \longrightarrow 0,$$

where $S^*(M \mid K+L) = S^*(M \mid K) + S^*(M \mid L)$ in $S^*(M)$. Using locality principle, one can check that $S^*(M \mid K+L)$ induces the same homology groups as $S^*(M \mid K \cup L)$. Thus, we get the long exact sequence

$$\cdots \to H^i_{\mathrm{cpt}}(M \mid K \cap L) \to H^i_{\mathrm{cpt}}(M \mid K) \oplus H^i_{\mathrm{cpt}}(M \mid L) \to H^i_{\mathrm{cpt}}(M \mid K \cup L) \to \dots .$$

By excision, we may replace these cohomology groups with the corresponding isomorphic cohomology groups

$$\cdots \to H^i_{\mathrm{cpt}}(U \cap V \mid K \cap L) \to H^i_{\mathrm{cpt}}(U \mid K) \oplus H^i_{\mathrm{cpt}}(V \mid L) \to H^i_{\mathrm{cpt}}(U \cup V \mid K \cup L) \to \dots ,$$

and taking colimit across K and L gives the result. ☐

We next state the following technical result.

Lemma 8.3.13.

If one define $[U \cup V]$, $[U]$, and $[V]$ by applying the inclusion map from $[U \cap V]$, then following diagram commutes

$$\begin{tikzcd}
\dots \arrow[r] & H^i_{\mathrm{cpt}}(U \cap V) \arrow[r] \arrow[d, "\frown [U\cap V]"] & H^i_{\mathrm{cpt}}(U) \oplus H^i_{\mathrm{cpt}}(V) \arrow[r] \arrow[d, "(\frown [U], \frown [V])"] & H^i_{\mathrm{cpt}}(U \cup V) \arrow[r] \arrow[d, "\frown [U\cup V]"] & \dots \\
\dots \arrow[r] & H_{n-i}(U \cap V) \arrow[r] & H_{n-i}(U) \oplus H_{n-i}(V) \arrow[r] & H_{n-i}(U \cup V) \arrow[r] & \dots
\end{tikzcd}$$

(where the bottom row is the normal Mayer-Vietoris).

▶ *Proof.* We have to check that the following diagram commutes for any $K \in \mathrm{cpt}(U)$ and $L \in \mathrm{cpt}(V)$, the following diagram commutes.

$$\begin{tikzcd}
\dots \arrow[r] & H^i(U \cap V \mid K \cap L) \arrow[r] \arrow[d, "\frown [U\cap V]"] & H^i(U \mid K) \oplus H^i(V \mid L) \arrow[r] \arrow[d, "(\frown [U], \frown [V])"] & H^i(U \cup V \mid K \cup L) \arrow[r] \arrow[d, "\frown [U\cup V]"] & \dots \\
\dots \arrow[r] & H_{n-i}(U \cap V) \arrow[r] & H_{n-i}(U) \oplus H_{n-i}(V) \arrow[r] & H_{n-i}(U \cup V) \arrow[r] & \dots
\end{tikzcd},$$

Then taking colimit across all (K, L) will give the desired exact sequence.

The two squares shown above clearly commutes by the projection formula (Proposition 8.3.2 (b)). What's problematic is the square with the boundary map. To do this, we start with an element in $H^i(U \cup V \mid K \cup L)$. This element was constructed as an element in $S^i(M \mid K+L)$, so it is of the form $a - b$, where $a \in S^i(U \cup V \mid K)$ and $b \in S^i(U \cup V \mid L)$ such that $\partial(a-b) = 0$.

We now compute the connecting homomorphism. Let us start with the top one. Recall that it is constructed from the following diagram of chain complexes.

$$\begin{tikzcd}
0 \arrow[r] & S^i(M \mid K \cap L) \arrow[r] \arrow[d, "\partial"] & S^i(M \mid K) \oplus S^i(M \mid L) \arrow[r] \arrow[d, "\partial"] & S^i(M \mid K+L) \arrow[r] \arrow[d, "\partial"] & 0 \\
0 \arrow[r] & S^{i+1}(M \mid K \cap L) \arrow[r] & S^{i+1}(M \mid K) \oplus S^i(M \mid L) \arrow[r] & S^{i+1}(M \mid K+L) \arrow[r] & 0
\end{tikzcd}$$

We have (a, b) is an element of the middle square of the top row, which is then mapped down to $(\partial a, \partial b)$. Then $\delta(a-b) = c$, where c is such that $\partial a = \partial b = c$.

Now, let's do the bottom one. We have an element $(a|_U \frown [U], b|_V \frown [V]) \in S^i(U) \oplus S^i(V)$. Therefore, we have that $\delta((a|_U \frown [U]) - (b|_V \frown [V])) = \gamma$ where γ is such that $\partial(a|_U \frown [U]) = i_* \gamma$ and $\partial(b|_V \frown [V]) = j_* \gamma$, where $i : U \cap V \to U$ and $j : U \cap V \to V$.

We have to show that $c|_{U\cap V} \frown [U \cup V] = \gamma$. To do this, note that

$$\begin{aligned}
\partial(a|_U \frown [U]) &= \partial a|_U \frown [U] + (-1)^i a|_U \frown (\partial[U]) \\
&= i_*(c|_{U\cap V} \frown [U \cap V]) + 0 = \gamma,
\end{aligned}$$

where $\partial a|_U \frown [U] = i_*(c|_{U\cap V} \frown [U \cap V])$ is by projection formula (Proposition 8.3.2 (b)), and $\partial[U] = 0$ since $[U]$ is a cycle. This implies that the square commutes. Similarly, $\partial(b \frown [V]) = j_*\gamma$, showing that this γ works. Thus, the square commutes. □

Corollary 8.3.14.

If Poincaré Duality is true for U, V, and $U \cap V$, then it is true for $U \cup V$.

▶ *Proof.* Apply five lemma on the diagram

$$\begin{tikzcd}
H^i_{\mathrm{cpt}}(U\cap V) \arrow[r] \arrow[d, "\sim"] & H^i_{\mathrm{cpt}}(U)\oplus H^i_{\mathrm{cpt}}(V) \arrow[r] \arrow[d, "\sim"] & H^i_{\mathrm{cpt}}(U\cup V) \arrow[r] \arrow[d] & H^{i+1}_{\mathrm{cpt}}(U\cap V) \arrow[r] \arrow[d, "\sim"] & H^{i+1}_{\mathrm{cpt}}(U)\oplus H^{i+1}_{\mathrm{cpt}}(V) \arrow[d, "\sim"] \\
H_{n-i}(U\cap V) \arrow[r] & H_{n-i}(U)\oplus H_{n-i}(V) \arrow[r] & H_{n-i}(U\cup V) \arrow[r] & H_{n-i-1}(U\cap V) \arrow[r] & H_{n-i-1}(U)\oplus H_{n-i-1}(V)
\end{tikzcd},$$

which is shown by the previous lemma to commute. □

▶ **Proof of Poincaré Duality III.** We now explain the proof of Theorem 8.3.9. First, we note that the statement is true for $\mathbb{R}^n$ by Example 8.3.10. Now, the steps are the following.

(a) First, Poincaré duality is true for a convex open subset $U \subseteq \mathbb{R}^n$. This follows from Example 8.3.10.

(b) Next, note that if $U_1, \dots, U_n$ are convex open subsets of $\mathbb{R}^n$, then Poincaré duality is true for $U_1 \cup \dots \cup U_n$. This follows from Lemma 8.3.13 and induction argument similar to the third step of the proof of Theorem 8.2.7.

(c) Prove that if $U_1 \subseteq U_2 \subseteq \dots$ is a chain of open subsets, and if Poincaré duality is true for $U_1, U_2, \dots$, then it is true for $U = U_1 \cup U_2 \cup \dots$. Note that we have a map
$$H^p_{\mathrm{cpt}}(U_i, R) \to H^p_{\mathrm{cpt}}(U_{i+1}, R)$$
because the second colimit is a colimit over a larger collection of compact sets (this uses Excision, Theorem 4.4.1). Thus, we deduce that
$$H^p_{\mathrm{cpt}}(U, R) = \operatorname*{colim}_i H^p_{\mathrm{cpt}}(U_i, R).$$
Moreover, by the same argument as in the fourth step of the proof of Theorem 8.2.7, we get that
$$H_{n-p}(U, R) = \operatorname*{colim}_i H_{n-p}(U_i, R).$$
Hence, we conclude that $H^p_{\mathrm{cpt}}(U, R) \simeq H_{n-p}(U, R)$ because they are colimit of isomorphic diagrams.

The same argument generalize to any (possibly uncountable) chains of open sets.

(d) Next, we note that if $U_1, U_2, \dots$ is a sequence of convex open sets of $\mathbb{R}^n$ such that Poincaré duality is true, then Poincaré duality is true for $U_1 \cup U_2 \cup \dots$ as well. This follows from applying (c) on $U_1 \subseteq U_1 \cup U_2 \subseteq U_1 \cup U_2 \cup U_3 \subseteq \dots$.

Thus, we deduce that Poincaré duality is true for any open sets of $\mathbb{R}^n$. This follows since any open set of $\mathbb{R}^n$ is a countable union of open balls, by picking every open ball with rational center and rational radius.

(e) At this point, we now know that Poincaré duality is true for any compact manifold M, because those are finite union of open sets.

To handle general manifolds, we use Zorn's lemma. Consider the poset
$$\mathcal{P} = \{U \subseteq M \text{ open}: \text{ Poincaré duality is true on } U\}.$$
From (c), we know that the hypothesis of Zorn's lemma (that any chain of $\mathcal{P}$ has an element in $\mathcal{P}$ above all of them) is satisfied, so $\mathcal{P}$ has a maximal element, say U. If $x \notin U$, we pick an open ball $V \ni x$ and then since $U \in \mathcal{P}$, $V \in \mathcal{P}$, and $U \cap V \in \mathcal{P}$ because it is an open subset of $V \subseteq \mathbb{R}^n$, we get that $U \cup V \in \mathcal{P}$, so U is not a maximal element, a contradiction. This completes the proof of Poincaré Duality.

§8.4 Fully Relative Poincaré Duality

In this section, we state (without proof) the fully relative version of Poincaré Duality.

§8.4.1 Fully Relative

There is an even more general version of Poincaré duality, in which we consider relative cohomology over arbitrary subset. However, the usual cohomology does not work here, so we need a different kind of cohomology to make the theorem work.

▶ Čech Cohomology

Definition 8.4.1.

For any manifold M and any compact subset $K \subseteq M$, define the **Čech cohomology** as

$$\check{H}^i(K, R) = \operatorname*{colim}_{\substack{U \text{ open} \\ K \subseteq U}} H^i(U, R).$$

Similarly, for any compact subsets $L \subseteq K \subseteq M$, we can define a relative version of Čech cohomology as

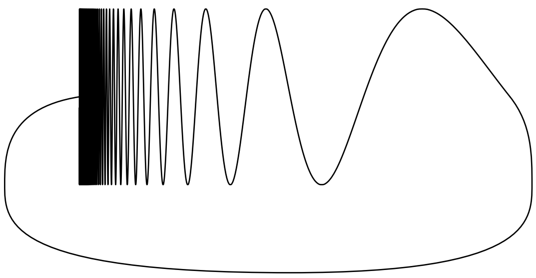
$$\check{H}^i(K, L, R) = \operatorname*{colim}_{\substack{U \supset K \\ V \supset L}} H^i(U, V, R).$$

Notice how strange this definition is in that it relies on the bigger manifold M. This is not the morally correct definition.

Čech cohomology satisfies all cohomology axioms; in particular, it satisfies long exact sequence property. However, it is not the same as normal cohomology.

Example 8.4.2.

Consider the **Warsaw circle** K embedded in $\mathbb{R}^2$, which connects the endpoints of the topologist sine curve (in Warning 1.3.11):



Without justifying details, we claim that $H^1(K, \mathbb{Z}) = H_1(K, \mathbb{Z}) = 0$, but $\check{H}^1(K, \mathbb{Z}) = \mathbb{Z}$.

Still, in the most sane cases, when K is a CW complex, Čech cohomology agrees with normal cohomology.

Lemma 8.4.3.

Suppose that $K \subseteq M$ safisfies the **regular neighborhood condition** that for any open set $U \supseteq K$, there exists an open set $U \supseteq V \supseteq K$ such that $K \hookrightarrow V$ is a homotopy equivalence, then

$$\check{H}^i(K, R) \simeq H^i(K, R)$$

for all i.

▶ *Proof.* We define the canonical map $\psi_U : H^i(U,R) \to H^i(K,R)$ by inducing it from inclusion. We check that this satisfies the universal property of colimits.

To do this, suppose we have a system of map $\phi_U : H^i(U,R) \to T$ for each open set $U \supseteq K$. We pick open set W such that $K \hookrightarrow W$ is a homotopy equivalence. Then our map is the composition of $\phi_W : H^i(W,R) \to T$ and the isomorphism $H^i(W,R) \to H^i(K,R)$. We have to check that the following diagram commutes for all $U \supseteq K$:

$$\begin{tikzcd}
 & H^i(K,R) \arrow[dr, leftrightarrow, "\simeq"] & \\
H^i(U,R) \arrow[ur] \arrow[dr, "\phi_U"'] & & H^i(W,R) \arrow[dl, "\phi_W"] \\
 & T &
\end{tikzcd}$$

To do this, we take $U \cap W \supseteq V \supseteq K$ such that $K \hookrightarrow V$ is a homotopy equivalence. Then the commutativity of the above follows from commutativity of this diagram

$$\begin{tikzcd}
 & & H^i(K,R) \arrow[d, leftrightarrow, "\simeq"'] \arrow[drr, leftrightarrow, "\simeq"] & & \\
H^i(U,R) \arrow[urr] \arrow[rr] \arrow[drr, "\phi_U"'] & & H^i(V,R) \arrow[rr, leftrightarrow, "\simeq"'] \arrow[d, "\phi_V"] & & H^i(W,R) \arrow[dll, "\phi_W"] \\
 & & T & &
\end{tikzcd}.$$

It is easy to see that this map is unique. □

Now, we can state the even more general version of Poincaré duality. The proof is similar to Poincaré Duality III (Theorem 8.3.9), although it is slightly more technical. See [Mil21] for more detail.

Theorem 8.4.4 (Poincaré Duality IV)**.**

Let M be a compact, R-oriented, n-dimensional manifold. Let $L \subseteq K$ be compact subsets. Then,

$$\check{H}^i(K,L,R) \simeq H_{n-i}(M-L, M-K, R)$$

When $K = M$ and $L = \emptyset$, the theorem above gives

$$H^i(M,R) \simeq H_{n-i}(M,R),$$

recovering Poincaré duality.

§8.4.2 Alexander Duality

Using this, we can deduce an interesting consequence. First, to simplify the result, we introduce **reduced homology group** $\widetilde{H}_i(X,R)$, defined as relative homology to a point

$$\widetilde{H}_i(X,R) = H_i(X, \text{point}, R) = \begin{cases} H_i(X,R) & i \geq 1 \\ H_0(X,R)/\mathbb{Z} & i = 0. \end{cases}.$$

This simplifies homology calculation because $\widetilde{H}_i(\text{point}) = 0$ always, so there is no edge case of $i = 0$.

Now, we state the theorem.

Theorem 8.4.5 (Alexander Duality).

For any compact subset $K \subsetneq S^n$ and any ring R, we have

$$\check{H}^i(K, R) \simeq \widetilde{H}_{n-i-1}(S^n - K, R).$$

▶ *Proof.* Take $M = S^n$ and apply Theorem 8.4.4 on $(K, \emptyset)$. We then have the diagram

$$\begin{tikzcd} & \check{H}^i(K,R) \arrow[d, "\sim"] & & \\ \underbrace{\widetilde{H}_{n-i}(S^n, R)}_{=0} \arrow[r] & \widetilde{H}_{n-i}(S^n, S^n - K, R) \arrow[r] & \widetilde{H}_{n-i-1}(S^n - K, R) \arrow[r] & \underbrace{\widetilde{H}_{n-i-1}(S^n, R)}_{=0} \end{tikzcd},$$

where the bottom row comes from long exact sequence of homology. Note that by homology of sphere, the left and right factors are 0. Thus, the composition

$$\check{H}^i(K, R) \longrightarrow H_{n-i}(S^n, S^n - K, R) \longrightarrow H_{n-i-1}(S^n - K, R)$$

is an isomorphism. □

§8.5 Lefschetz Fixed Point Theorem

In this section, we explore another application of Poincaré Duality: Lefschetz Fixed Point Theorem (Theorem 8.5.5). This theorem relates the number of fixed point with trace of maps in homology groups. Despite being first discovered as topological fact, this theorem motivates the study of algebraic varieties via cohomology, eventually leading to theorems outside topology such as the proof of Weil's conjectures.

§8.5.1 Intersection Pairing

First, we set up notations. For simplicity, we will work over $\mathbb{Q}$. The main advantage for this is that we have isomorphisms

$$\times : H^*(X, \mathbb{Q}) \otimes_{\mathbb{Q}} H^*(Y, \mathbb{Q}) \to H^*(X \times Y, \mathbb{Q})$$
$$\times : H_*(X, \mathbb{Q}) \otimes_{\mathbb{Q}} H_*(Y, \mathbb{Q}) \to H_*(X \times Y, \mathbb{Q}).$$

The first one is the cohomology cross product. The second one is known as the **homology cross product**, which can be obtained by inverting the Alexander-Whitney map.

Throughout this section, let M be a compact, connected, $\mathbb{Q}$-oriented n-manifold.

For each $\alpha \in H_{n-i}(M, \mathbb{Q})$, let $\alpha^* \in H^i(M, \mathbb{Q})$ be the corresponding cohomology class due to Poincaré duality. In particular, α^* is such that the cap product $\alpha^* \frown [M]$ is a.

Definition 8.5.1 (Intersection pairing).

For each $\alpha \in H_{n-i}(M, \mathbb{Q})$ and $\beta \in H_{n-j}(M, \mathbb{Q})$, we define the **intersection pairing**

$$\alpha \cdot \beta = (\alpha^* \smile \beta^*) \frown [M] \quad \in H_{n-(i+j)}(M, \mathbb{Q})$$

This is also equal to $\alpha^* \frown \beta$ by Proposition 8.3.2 (a).

We also observe that since cup product is graded comutative, we have

$$\alpha \cdot \beta = (-1)^{ij}(\beta \cdot \alpha). \tag{8.3}$$

Note also that if $i + j = n$, then the intersection pairing is a perfect pairing by Poincaré duality.

If A and B are submanifolds of M of dimension $n - i$ and $n - j$, then we can consider their Poincaré duality fundamental class $[A] \in H_{n-i}(M, \mathbb{Q})$ and $[B] \in H_{n-j}(M, \mathbb{Q})$ as representatives of the manifolds. Under favorable conditions (that A and B intersects traversely), $A \cap B$ is a submanifold of dimension $n - (i + j)$, and we in fact have

$$[A] \cdot [B] = [A \cap B].$$

We will not define "intersects transversely" precisely nor prove the above fact, since it requires differential geometry. However, we note the following small proposition.

Proposition 8.5.2.

If $A \cap B = \emptyset$, then $[A] \cdot [B] = 0$.

▶ *Proof.* Let $\alpha = [A]$ and $\beta = [B]$. Note by the construction of Poincaré duality, α^* comes from a cohomology class in $H^i(M, M - A)$. Thus, when it is evaluate on (the faces of) β, which is contained in $M - A$, we will get 0. Thus, $\alpha^* \frown \beta = 0$. □

Example 8.5.3.

Let $M = \mathbb{CP}^2$. Then, recall that

$$H_1(\mathbb{CP}^2, \mathbb{Q}) \simeq H^1(\mathbb{CP}^2, \mathbb{Q}) \simeq \begin{cases} \mathbb{Q} & i = 0, 2, 4 \\ 0 & \text{otherwise.} \end{cases}$$

Now, suppose that f and g are homogeneous polynomial in $\mathbb{C}[x, y, z]$, and take submanifolds

$$A = \{(x : y : z) \in \mathbb{CP}^2 : f(x, y, z) = 0\},$$
$$B = \{(x : y : z) \in \mathbb{CP}^2 : g(x, y, z) = 0\}.$$

Under favorable condition, A and B will have dimension 2. and their fundamental class is determined by degrees.

$$[A] = \deg f \in H_2(\mathbb{CP}^2, \mathbb{Q}), \quad [B] = \deg g \in H_2(\mathbb{CP}^2, \mathbb{Q}).$$

Thus, their intersection pairing is

$$[A \cap B] = [A] \cdot [B] = (\deg f)(\deg g).$$

This is a topological analogue of **Bezout's theorem**: curves with degree d and e meet at de points.

§8.5.2 Lefschetz Fixed Point Theorem

Again, let M be a compact, connected, $\mathbb{Q}$-oriented, n-dimensional manifold.

Now, for any continuous map $f : M \to M$, we define

$$\Gamma_f = \{(x, f(x)) : x \in M\}$$
$$\Delta = \{(x, x) : x \in M\}.$$

Then, the **Lefschetz number** of a map f is defined as

$$L(f) = [\Gamma_f] \cdot [\Delta] \in H_0(M \times M, \mathbb{Q}).$$

Remark 8.5.4.

Under favorable conditions, $L(f)$ is a signed count of the number of fixed points of f. The sign is determined by the orientation of the manifold.

Theorem 8.5.5 (Lefschetz Fixed Point Theorem)**.**

For any continuous map $f : M \to M$, we have

$$\begin{aligned} L(f) &= \sum_{i=0}^{n} (-1)^i \operatorname{tr}(f_* : H_i(M, \mathbb{Q}) \to H_i(M, \mathbb{Q})) \\ &= \sum_{i=0}^{n} (-1)^i \operatorname{tr}(f^* : H^i(M, \mathbb{Q}) \to H^i(M, \mathbb{Q})). \end{aligned}$$

Combining with Proposition 8.5.2, we have the following corollary, which is the most useful form of the theorem.

Corollary 8.5.6.

If $L(f) \neq 0$, then f has a fixed point.

To prove this theorem, we need several lemmas. The first one shows that cap product commutes with cross product.

Lemma 8.5.7 (Cap product commutes with cross product)**.**

Let X and Y be topological spaces. For any $\alpha \in H_*(X, R)$, $\beta \in H_*(Y, R)$, $a \in H^*(X, R)$, and $b \in H^*(Y, R)$, we have

$$(a \frown \alpha) \times (b \frown \beta) = (-1)^{\deg b \deg \alpha} (a \times \alpha) \frown (b \times \beta).$$

▶ *Proof.* First, we observe that the cup product map $\bullet \frown \alpha$ is a composition of the map

$$S_*(X) \xrightarrow{\Delta_X} S_*(X) \otimes_R S_*(X) \xrightarrow{1 \otimes a} S_*(X),$$

where $\Delta_X : X \to X \times X$ is the diagonal map.

It suffices to show that each step in this map commutes. For the first map (the Δ part), note that the following diagram commutes up to chain homotopy (by acyclic model).

$$\begin{tikzcd}
S_*(X) \otimes S_*(Y) \arrow[r, "\Delta_X \times \Delta_Y"] \arrow[dd, "\times"] & S_*(X) \otimes S_*(X) \otimes S_*(Y) \otimes S_*(Y) \arrow[r, "1\otimes a \otimes 1 \otimes b"] \arrow[d, "T"] & S_*(X) \otimes S_*(Y) \arrow[d, dashed, "\text{sign}"] \\
 & S_*(X) \otimes S_*(Y) \otimes S_*(X) \otimes S_*(Y) \arrow[r, "1 \otimes 1 \otimes a \otimes b"] \arrow[d, "\times \otimes \times"] & S_*(X) \otimes S_*(Y) \arrow[d, "\times"] \\
S_*(X \times Y) \arrow[r, "\Delta_{X \times Y}"] & S_*(X \times Y) \otimes S_*(X \times Y) \arrow[r, "1 \otimes (a \times b)"] & S_*(X \times Y)
\end{tikzcd}$$

Here all maps except the dashed arrow are chain maps, and T is the signed swap map used in the proof that cup product is graded commutative (Theorem 7.3.4 (c)). The "sign" map is the sign correction resulting from T, sending $\alpha \otimes \beta$ to $(-1)^{\deg b \deg \alpha}(\alpha \otimes \beta)$.

The left square commutes up to chain homotopy due to acyclic model argument (since they are all chain maps). The two right squares can be checked directly to commute at the chain level.

When we take homology, all chain maps descend down to map between homology groups. The sign map also does because it still sends a boundary to a boundary. At the homology level, every square commutes. Now, if we start from $a \otimes b$, then following the top-right path gives $(-1)^{\deg b \deg \alpha}(a \frown$

$\alpha) \times (b \frown \beta)$. Following the bottom-left path gives $(a \times \alpha) \frown (b \times \beta)$. Commutativity of the whole diagram gives the result. □

Lemma 8.5.8.

For any

$$\alpha \in H_i(M, \mathbb{Q}), \quad \beta \in H_{n-i}(M, \mathbb{Q}), \quad \gamma \in H_j(M, \mathbb{Q}), \quad \delta \in H_{n-j}(M, \mathbb{Q}),$$

we have

$$(\alpha \times \beta) \cdot (\gamma \times \delta) = \begin{cases} (-1)^{n-i}(\alpha \cdot \gamma)(\beta \cdot \delta) & \text{if } i + j = n \\ 0 & \text{otherwise.} \end{cases}$$

▶ *Proof.* From Lemma 8.5.7 (with $\alpha = \beta = [M]$), we have that for any $\gamma, \delta \in H_*(M)$, we have

$$(\gamma \times \delta)^* = (-1)^{n \cdot \deg \gamma}(\gamma^* \times \delta^*). \tag{8.4}$$

Therefore,

$$\begin{aligned}
(\alpha \times \beta) \cdot (\gamma \times \delta) &= (-1)^{n(i+j)}((\alpha^* \times \beta^*) \smile (\gamma^* \times \delta^*)) \frown [M \times M] \\
&= (-1)^{n(i+j)}(-1)^{j(n-i)}((\alpha^* \smile \gamma^*) \times (\beta^* \smile \delta^*)) \frown ([M] \times [M]) \\
& \hspace{20em} \text{(Lemma 7.3.5)} \\
&= (-1)^{j(n-i)}((\alpha^* \smile \gamma^*) \frown [M]) \times ((\beta^* \smile \delta^*) \frown [M]) \\
&= (-1)^{j(n-i)}((\alpha \cdot \gamma) \smile (\beta \cdot \delta)). \qquad \text{(Lemma 8.5.7)}
\end{aligned}$$

We have three cases.

- If $i + j > n$, then $\deg(\alpha^* \times \gamma^*) > n$, so $\alpha^* \smile \gamma^* = 0$, implying that $(\alpha \times \beta) \cdot (\gamma \times \delta) = 0$.
- If $i + j < n$, then $\deg(\beta^* \times \delta^*) > n$, so $\beta^* \smile \delta^* = 0$, implying that $(\alpha \times \beta) \cdot (\gamma \times \delta) = 0$.
- If $i + j = n$, then $\alpha \cdot \gamma$ and $\beta \cdot \delta$ are both scalars, so $(\alpha \times \beta) \cdot (\gamma \times \delta) = (\alpha \cdot \gamma)(\beta \cdot \delta)$, implying the result. □

The next lemma gives the formula for intersecting Γ_f and $\alpha \times \beta$. Intuitively, for any submanifolds A and B with $\alpha = [A]$ and $\beta = [B]$, we have that

$$\Gamma_f \cap (A \times B) = \{(a, b) \in A \times B : b = f(a)\}.$$

The left side corresponds to $\Gamma_f \cdot (\alpha \times \beta)$, while the right side corresponds to $f_*(\alpha) \cdot \beta$. The lemma then asserts that these are equal formally.

Lemma 8.5.9.

For any $\alpha \in H_i(M, \mathbb{Q})$ and $\beta \in H_{n-i}(M, \mathbb{Q})$, we have

$$[\Gamma_f] \cdot (\alpha \times \beta) = f_*(\alpha) \cdot \beta.$$

▶ *Proof.* Let $j : M \to M \times M$ be the map that sends x to $(x, f(x))$. Thus, $j_*([M]) = [\Gamma_f]$. Note that

$$\begin{aligned}
[\Gamma_f] \cdot (\alpha \times \beta) &= (-1)^n \left((\alpha \times \beta)^* \cdot [\Gamma_f]\right) && \text{(using (8.3))} \\
&= (\alpha^* \times \beta^*) \frown [\Gamma_f] && \text{(using (8.4))} \\
&= (\alpha^* \times \beta^*) \frown j_*([M]) && \text{(since } [\Gamma_f] = j_*([M])\text{)} \\
&= j_*\Big(j^*(\alpha^* \times \beta^*) \frown [M]\Big) && \text{(Projection formula, Proposition 8.3.2 (b))} \\
&= j_*\Big(\Delta^*(\mathrm{id} \times f)^*(\alpha^* \times \beta^*) \frown [M]\Big) && \text{(since } j = (\mathrm{id} \times f) \circ \Delta\text{)} \\
&= j_*\Big(\Delta^*(\alpha^* \times f^*(\beta^*)) \frown [M]\Big) \\
&= j_*\Big((\alpha^* \smile f^*(\beta^*)) \frown [M]\Big) && \text{(definition of cup product)} \\
&= (-1)^{i(n-i)} j_*\Big((f^*(\beta^*) \smile \alpha^*) \frown [M]\Big) && \text{(cup product is graded commutative)} \\
&= (-1)^{i(n-i)} j_*(f^*(\beta^*) \frown \alpha) && \text{(Proposition 8.3.2 (a))}
\end{aligned}$$

Composing with $\pi : M \times M \to M$ by projection onto the second coordinate

$$\begin{aligned}
\pi_*([\Gamma_f] \cdot (\alpha \times \beta)) &= (-1)^{i(n-i)} f_*(f^*(\beta^*) \frown \alpha) && \text{(since } \pi \circ j = f\text{)} \\
&= (-1)^{i(n-i)} (\beta^* \frown f_*(\alpha)) && \text{(Projection formula, Proposition 8.3.2 (b))} \\
&= (-1)^{i(n-i)} (\beta \cdot f_*(\alpha)) \\
&= f_*(\alpha) \cdot \beta, && \text{(using (8.3))}
\end{aligned}$$

so viewing both sides as integers, we get the desired result. □

Lemma 8.5.10 (Fundamental class of Δ).

We have
$$[\Delta] = \sum_k (-1)^{\deg e_k} (e_k \times e_k^\vee),$$
where $\{e_k\}$ is a basis of $H_*(M, \mathbb{Q})$ and $\{e_k^\vee\}$ is dual basis with respect to the intersection pairing (so $e_k \cdot e_k^\vee = 1$ for all k).

▶ *Proof.* By Kunneth's theorem, $\{e_i^\vee \times e_j\}$ is a basis for $H^i(M \times M, \mathbb{Q})$. If i and j are such that $\deg e_i^\vee + \deg e_j = n$, we have

$$\begin{aligned}
\left(\sum_k (-1)^{\deg e_k} (e_k \times e_k^\vee)\right) \cdot (e_i^\vee \times e_j) &= \sum_k (-1)^{\deg e_k} (e_k \times e_k^\vee) \cdot (e_i^\vee \times e_j) \\
&= (-1)^n \sum_{\substack{k \\ \deg e_k = \deg e_j}} (e_k \cdot e_i^\vee)(e_k^\vee \cdot e_j) && \text{(Lemma 8.5.8)} \\
&= (-1)^n (e_j \cdot e_i^\vee) && \text{(All terms vanish except } k = j\text{)} \\
&= e_i^\vee \cdot e_j && \text{(using (8.3))} \\
&= [\Delta] \cdot (e_i^\vee \times e_j) && \text{(Lemma 8.5.9 with } f = \mathrm{id}\text{)}
\end{aligned}$$

Since this holds for any $e_i^\vee \times e_j$ in the basis, we get that $[\Delta] = \sum_k e_k \times e_k^\vee$. □

▶ *Proof of Theorem 8.5.5.* We compute

$$\begin{aligned}
[\Gamma_f]\cdot[\Delta] &= [\Gamma_f]\cdot\sum_k e_k\times e_k^\vee && \text{(Lemma 8.5.10)}\\
&= \sum_k [\Gamma_f]\cdot(e_k\times e_k^\vee)\\
&= \sum_k (-1)^{\deg e_k} f_*(e_k)\cdot e_k^\vee && \text{(Lemma 8.5.9)}\\
&= \sum_{i=0}^n (-1)^i \operatorname{tr}(f_*: H_i(M,\mathbb{Q})\to H_i(M,\mathbb{Q}))
\end{aligned}$$

The last step follows from the definition of trace. □

Remark 8.5.11 (Lefschetz Fixed Point Theorem in other contexts).

Although Lefschetz's Fixed Point theorem was first discovered in the topological context, it has been applied in other contexts. Most famously, it has been used to prove the **Weil's conjectures**, which is a conjecture that counts the number of points of varieties over finite field $\mathbb{F}_q$.

More specifically, given a projective variety over $\mathbb{F}_q$, which can be viewed as a subset of $\mathbb{P}^n(\overline{\mathbb{F}}_q)$, a point $(x_0 : x_1 : \cdots : x_n) \in \mathbb{P}^n(\overline{\mathbb{F}}_q)$ has coordinates in $\mathbb{F}_q$ if and only if it is a fixed point of the **Frobenius map**

$$(x_0 : x_1 : \cdots : x_n) \mapsto (x_0^q : x_1^q : \cdots : x_n^q).$$

In particular, the proof of Weil conjectures uses a variant of Lefschetz's Fixed Point Theorem on varieties over finite fields to study the number of fixed points of the Frobenius map.

However, before we can do that, we need to develop a cohomology theory that works for variety over any field K. This theory is called **étale cohomology**, which agrees with singular cohomology when $K = \mathbb{C}$.

§8.6 Problems

Problem 8.A. Prove that $S^1 \vee S^2 \vee S^3$ is not homotopy equivalent to a compact manifold.

Problem 8.B. Let M be a compact manifold. Prove that $H_i(M, R)$ is a finitely-generated R-module for all i.

Problem 8.C. Prove that the cohomology ring $H^*(\mathbb{CP}^n, \mathbb{Z})$ is $\mathbb{Z}[x]/(x^{n+1})$ where $\deg x = 2$.

Problem 8.D (Hopf Map). Consider two maps from $S^3 \to S^2$. The first one f_1 sends every point in S^3 to $(1, 0, 0) \in S^2$. The second one is the **Hopf map** defined by

$$\begin{aligned}
f_2 : S^3 &\to \mathbb{CP}^1 \simeq S^2\\
(a, b, c, d) &\mapsto (a + bi : c + di).
\end{aligned}$$

The goal is to show that f_1 and f_2 are not homotopic.

(a) For $i = 1, 2$, let P_i be the CW complex obtained by attaching a 4-cell to S^2 via f_i. Prove that if f_1 and f_2 are homotopic, then P_1 and P_2 are homotopy equivalent.

(b) Explicitly describe P_1 and P_2, and show that they are not homotopy equivalent.

For any compact, connected, and $\mathbb{Z}$-orientable n-manifold M, the **degree** of a map $f : M \to M$ is an integer k such that the induced map $H^n(f, \mathbb{Z}) : H^n(M, \mathbb{Z}) \to H^n(M, \mathbb{Z})$ is multiplication by k.

Problem 8.E (degree $\approx$ number of preimages)**.** Let M be a compact, connected, and $\mathbb{Z}$-orientable n-manifold. Let $f : M \to M$ be a continuous map, and let $x \in M$. Suppose that there exists a neighborhood $U \ni x$ such that $f^{-1}U = \coprod_{i=1}^{k} U_i$ such that $f|_{U_i}$ induces an isomorphism $U_i \to U$. (In other words, $f^{-1}U$ is a disjoint union of k copies of U.) Prove that f has degree $\pm k$.

Problem 8.F. (a) Prove that degree of any map $f : \mathbb{CP}^n \to \mathbb{CP}^n$ must be a perfect n-th power.

(b) Conversely, for any (not necessarily positive) integer k, given an example of a map with degree k^n.

Problem 8.G. (a) Prove that if n is even, then any map $f : \mathbb{CP}^n \to \mathbb{CP}^n$ must have a fixed point.

(b) Conversely, for any odd n, give an example of a map $f : \mathbb{CP}^n \to \mathbb{CP}^n$ with no fixed points.

Problem 8.H. Prove that any non-orientable 2-manifold (e.g., Klein bottle and $\mathbb{RP}^2$) cannot be embedded into a closed subset of $\mathbb{R}^3$.

Problem 8.I. A **division algebra** structure on $\mathbb{R}^n$ consists of a multiplication map

$$m : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$$

that is $\mathbb{R}$-bilinear and for all $a \neq 0 \in \mathbb{R}^n$, the maps $x \mapsto m(a, x)$ and $x \mapsto m(x, a)$ are isomorphisms. The goal of this problem is to show that a division algebra exists only if n is a power of two.

(a) Let S^{n-1} be a subset of unit vectors in $\mathbb{R}^n$. Define the map

$$\begin{aligned} h : S^{n-1} \times S^{n-1} &\to S^{n-1} \\ (x, y) &\mapsto \frac{m(x, y)}{|m(x, y)|}. \end{aligned}$$

Prove that h is continuous and induces a map $\overline{h} : \mathbb{RP}^{n-1} \times \mathbb{RP}^{n-1} \to \mathbb{RP}^{n-1}$.

(b) For a fixed $a \in \mathbb{RP}^{n-1}$, observe that $\overline{h}$ induces an isomorphism when restricted to either $\mathbb{RP}^{n-1} \times \{a\}$ or $\{a\} \times \mathbb{RP}^{n-1}$. Using this, describe where the generator of $H^*(\mathbb{RP}^{n-1}, \mathbb{F}_2)$ is sent to in the map $H^*(\overline{h}, \mathbb{F}_2) : H^*(\mathbb{RP}^{n-1}, \mathbb{F}_2) \to H^*(\mathbb{RP}^{n-1} \times \mathbb{RP}^{n-1}, \mathbb{F}_2)$.

(c) Derive a contradiction if n is not a power of two. You may find Lucas' theorem (about the remainder of binomial coefficients modulo prime number) helpful.

A Category Theory

§A.1 Category, Functors, and Natural Transformations

In order to think about infinitely-generated abelian groups such as the one appearing in the definition of singular homology, we first introduce the language of category theory, which is a general framework that unifies all mathematical structures like groups, vector space, topological spaces, etc.

§A.1.1 Category

Definition A.1.1 (Category).

A **category** $\mathscr{C}$ consists of

- a class of **objects** $\operatorname{obj}\mathscr{C}$;
- for each $X, Y \in \operatorname{obj}\mathscr{C}$, a set of **morphisms** $\operatorname{Hom}_{\mathscr{C}}(X, Y)$;
- for each $X \in \operatorname{obj}\mathscr{C}$, the identity map $1_X \in \operatorname{Hom}_{\mathscr{C}}(X, X)$; and
- the composition map $\operatorname{Hom}_{\mathscr{C}}(X, Y) \times \operatorname{Hom}_{\mathscr{C}}(Y, Z) \to \operatorname{Hom}_{\mathscr{C}}(X, Z)$ for each $X, Y, Z \in \operatorname{obj}\mathscr{C}$, often written as $(f, g) \mapsto g \circ f$.

These data must satisfy the following.

- **Associative.** $f \circ (g \circ h) = (f \circ g) \circ h$ for all compatible f, g, h.
- **Identity.** For any $f : X \to Y$, we have $1_Y \circ f = f \circ 1_X = f$.

Example A.1.2 (Every mathematical structure is a category).

This definition captures various structure that we have seen so far; for example,

- category $\mathbf{Sets}$ with objects sets and morphisms being functions between two sets;
- category $\mathbf{Top}$ of topological spaces and morphisms being continuous functions;
- category $\mathbf{Ab}$ of abelian groups and morphisms being group homomorphisms;
- category $\mathbf{Vect}_{\mathbb{R}}$ of vector spaces and morphisms being linear transformations.

Remark A.1.3.

$\operatorname{obj}\mathscr{C}$ is not necessarily a set. For example, objects category $\mathbf{Sets}$ do not form a set. However, no one cares, perhaps except set theorists.

If $\mathscr{C}$ is a category, we write $X \in \mathscr{C}$ to mean $X \in \operatorname{obj}\mathscr{C}$ and write $f : X \to Y$ to mean $f \in \operatorname{Hom}_{\mathscr{C}}(X, Y)$.

The definition of isomorphism is what you might expect.

Definition A.1.4 (Isomorphism)**.**

In category $\mathscr{C}$, a morphism $f : X \to Y$ is an **isomorphism** if there exists $g : Y \to X$ such that $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$.

Example A.1.5 (Isomorphism)**.**

We have the following examples.

- An isomorphism in **Sets** is a bijection.
- An isomorphism in **Top** is a homeomorphism (i.e., continuous map with continuous inverse). Note that this is not the same as continuous bijection.

Proposition A.1.6.

Suppose that $f : X \to Y$ is an isomorphism, then the inverse $g : Y \to X$ is unique.

▶ *Proof.* If $g' : Y \to X$ is the second inverse, then

$$g' = (g \circ f) \circ g' = g \circ (f \circ g') = g.$$

□

§A.1.2 Functor and Natural Transformations

When we encounter a new mathematical structure, we usually also study maps between them. Maps between categories are called functor.

Definition A.1.7 (Functor)**.**

A **functor** $F : \mathscr{C} \to \mathscr{D}$ between two categories $\mathscr{C}$ and $\mathscr{D}$ consists of

- an assignment $F : \operatorname{obj}\mathscr{C} \to \operatorname{obj}\mathscr{D}$ and
- for all $X, Y \in \operatorname{obj}\mathscr{C}$, a function

$$F : \operatorname{Hom}_{\mathscr{C}}(X, Y) \to \operatorname{Hom}_{\mathscr{D}}(F(X), F(Y))$$

that satisfies

- $F(1_X) = 1_{F(X)}$ for all $X \in \operatorname{obj}\mathscr{C}$; and
- if $g \circ f$ makes sense in $\mathscr{C}$, then $F(g \circ f) = F(g) \circ F(f)$.

It is common to write the induced map $F(f)$ as f_*.

Let's now discuss some examples of functors.

Example A.1.8 (Free abelian group is a functor)**.**

There is a functor $\mathrm{Free} : \mathbf{Sets} \to \mathbf{Ab}$ sending a set to the free abelian group generated by that set. This is asserting that any map between two sets induces a group homomorphism of the corresponding free abelian groups.

We now define what is a **natural transformation**.

Definition A.1.9 (Natural Transformation).

Let F and G be two functors from $\mathscr{C}$ to $\mathscr{D}$. A **natural transformation** $\Theta : F \to G$ consists of morphisms $\Theta_X : F(X) \to G(X)$ for all $X \in \mathscr{C}$ such that the following diagram commutes

$$\begin{array}{ccc} F(X) & \xrightarrow{\Theta_X} & G(X) \\ \downarrow{\scriptstyle F(f)} & & \downarrow{\scriptstyle G(f)} \\ F(Y) & \xrightarrow{\Theta_Y} & G(Y) \end{array}$$

A natural transformation Θ is called a **natural isomorphism** if each of Θ_X is an isomorphism for all $X \in \operatorname{obj}\mathscr{C}$.

§A.1.3 Opposite Category

Sometimes, we want to deal with functors that is not quite a functor because it reverses the arrow.

As an example, given $V \in \mathbf{Vect}_{\mathbb{R}}$, there is a **dual vector space**

$$V^\vee = \{\text{linear map } V \to \mathbb{R}\}.$$

If $T : V \to W$ is a linear map in $\mathbf{Vect}_{\mathbb{R}}$, there is an induced map $T^* : W^\vee \to V^\vee$. The map is defined as follows, given $f \in W^\vee$, i.e., $f : W \to \mathbb{R}$, then $T^*(f) = f \circ T$. This is not a functor. However, there is a useful notion that allows us to deal with a *functor that reverses an arrow*.

Definition A.1.10 (Opposite Category).

Given a category $\mathscr{C}$, the **opposite category** $\mathscr{C}^{\text{op}}$ is a category with the same objects as $\mathscr{C}$, but

$$\operatorname{Hom}_{\mathscr{C}^{\text{op}}}(X, Y) = \operatorname{Hom}_{\mathscr{C}}(Y, X).$$

The composition law is defined by $(f \circ g)^{\text{op}} = g^{\text{op}} \circ f^{\text{op}}$.

Example A.1.11.

The dual space above is a functor $\bullet^\vee : \mathbf{Vect}_{\mathbb{R}} \to \mathbf{Vect}_{\mathbb{R}}^{\text{op}}$.

Remark A.1.12.

If we have a functor $F : \mathscr{C} \to \mathscr{D}^{\text{op}}$, then we typically write the induced map $F(f)$ as f^*. In particular,

- we put the asterisk on the subscript if the functor preserves the arrow; and
- we put the asterisk on the superscript if the functor reverses the arrow.

There is no reason for this other than it is a standard convention. However, this convention is common enough that you will get called out if you put the asterisk incorrectly.

We now give the classic example of natural transformation.

Example A.1.13.

There is a natural map from $\phi : V \to (V^\vee)^\vee$ defined as follows: an element $v \in V$ is sent to the "evaluation at v map" $\mathrm{ev}_v : V^\vee \to \mathbb{R}$ by $\mathrm{ev}_v(f) = f(v)$ for all $f : V \to \mathbb{R}$.

We now explain what it means by this map is natural. We note that there is a functor $\mathbf{Vect}_\mathbb{R} \to \mathbf{Vect}_\mathbb{R}$ given by $V \mapsto (V^\vee)^\vee$, obtained by applying the dual functor twice. Then the map above provides a natural transformation between the identity functor and the double dual functor. To see this, we have to check that the following diagram commutes for all linear maps $T : V \to W$.

$$\begin{tikzcd} V \arrow[r, "\phi"] \arrow[d, "T"] & (V^\vee)^\vee \arrow[d, "(T^*)^*"] \\ W \arrow[r, "\phi"] & (W^\vee)^\vee \end{tikzcd}$$

Checking this involves going through the definitions, and you might prefer to write it out yourself rather than following what's written below. Given $v \in V$, if we go down first and go right, we get $\mathrm{ev}_{T(v)} : W^\vee \to \mathbb{R}$. If we go right first, we get ev_v, and we have to figure out what this does under $(T^*)^*$.

We have that $(T^*)^*(\mathrm{ev}_v) = \mathrm{ev}_v \circ T^*$. For any $f \in W^\vee$, we have that $(T^*)^*(\mathrm{ev}_v)$ takes f to $\mathrm{ev}_v(T^*(f)) = \mathrm{ev}_v(f \circ T) = f(T(v))$. Thus, $(T^*)^*(\mathrm{ev}_v) = \mathrm{ev}_{T(v)}$, as desired.

§A.2 Categorical Constructions

§A.2.1 Universal Properties

Universal properties (also known as **universal mapping properties** or **mapping properties**) determine an object up to isomorphism. They are of a very specific format, which is best seen through examples.

▶ **Motivating Example: Product.** Let $\mathscr{C}$ be a category with objects M and N. Then, the product $M \times N$ is an object P, equipped with morphisms $\mu : P \to M$ and $\nu : P \to N$, such that the following universal property holds: "for any object P' and maps $\mu' : P' \to M$ and $\nu' : P' \to N$

$$\begin{tikzcd} P' \arrow[dr, dashed] \arrow[drr, "\mu'"] \arrow[ddr, "\nu'"'] & & \\ & P \arrow[r, "\mu"'] \arrow[d, "\nu"] & M \\ & N & \end{tikzcd},$$

there exists unique morphism $\phi : P' \to P$ (in dashed arrow) that makes the diagram above commute."

Let's suppose that $\mathscr{C}$ is the category **Sets** (objects are sets, and morphisms are any function between two sets). Given sets M and N, why does the definition above agree with the usual product

$$P = \{(m, n) : m \in M, n \in N\}?$$

First, the morphisms μ and ν are defined by $\mu((m, n)) = m$ and $\nu((m, n)) = n$. Now, let us verify that the above property holds. Suppose that there exists P', μ', and ν' as above. Then, $\phi : P' \to P$ is uniquely determined to be $\phi(p) = (\mu'(p), \nu'(p))$. Thus, P satisfy the above universal property.

Now, we claim that **any two products satisfying the universal property above are isomorphic**. To prove this, suppose that we have two products P_1 and P_2:

$$
\begin{tikzcd}
P_1 \arrow[r, "\mu_1"'] \arrow[d, "\nu_1"] & M \\
N &
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
P_2 \arrow[r, "\mu_2"'] \arrow[d, "\nu_2"] & M \\
N &
\end{tikzcd}
.
$$

Then, by letting P_2 play the role of P and P_1 play the role of P' gives a morphism $\phi_1 : P_1 \to P_2$ making the following diagram commute.

$$
\begin{tikzcd}
P_2 \arrow[dr, bend left, "\phi_2"] \arrow[drr, bend left, "\mu_2"] \arrow[ddr, bend right, "\nu_2"'] & & \\
& P_1 \arrow[ul, bend left, "\phi_1"] \arrow[r, "\mu_1"'] \arrow[d, "\nu_1"] & M \\
& N &
\end{tikzcd}
.
$$

Similarly, there exists a morphism $\phi_2 : P_2 \to P_1$ that makes the diagram above commute.

Now, we plug in $P = P' = P_1$. Then, there exists a unique morphism $P_1 \to P_1$ that commutes with both μ_1 and ν_1. However, we can name two such morphisms: $\phi_2 \circ \phi_1$ and the identity map. Thus, $\phi_2 \circ \phi_1 = \mathrm{id}_{P_1}$. Similarly, $\phi_1 \circ \phi_2 = \mathrm{id}_{P_2}$, so P_1 and P_2 are isomorphic.

Notice how this argument does not use the nature of our category $\mathscr{C}$ at all. Thus, if one changes category $\mathscr{C}$ to other categories (e.g., **Rings** or R-**Mod**), then the exact same argument shows that any two products are isomorphic. What's left, however, is to show that product exists for each category that you want to define a product.

▶ **Universal Property in General** In general, the recipe for defining an object through universal properties is as follows.

1. Define the universal property.
2. Prove that this property characterizes an object up to an isomorphism (a routine process).
3. Give a construction of an object (possibly difficult).

In these notes, we will soon see examples (e.g., tensor product, Section 6.3) where Step 3. is nontrivial, and therefore, the construction of the object itself makes it very hard to prove properties. When such situation arises, we often forget about Step 3. and use universal properties to deduce properties about that object.

▶ **Free Objects** We now define free groups, free abelian groups, etc. through universal properties.

Definition A.2.1.

Let $\mathscr{C}$ be a category whose objects have a set structure (e.g., groups, abelian groups, R-modules, or topological spaces). Given any set X, the **free object** is the object $F(X) \in \mathscr{C}$, together with a map (of sets) $X \to F(X)$ such that for any object $C \in \mathscr{C}$ and a map (of sets) $X \to C$, there exists morphism $f \in \mathrm{Hom}_{\mathscr{C}}(C, F(X))$ such that the following diagram commutes

$$
\begin{tikzcd}
X \arrow[rr] \arrow[dr] & & F(X) \\
& C \arrow[ur, "f"] &
\end{tikzcd}
.
$$

It is not very hard to see that free object is unique up to isomorphism.

We now give a construction for free objects in various category.

Example A.2.2 (Free R-modules)**.**

Given a set X, the free R-modules $R\{X\}$ has elements of the form $\sum_{i=1}^{n} r_i x_i$ where $x_i \in X$ and $r_i \in R$.

To see that this satisfies the universal property, suppose that M is an R-module, and we have a map $\phi : X \to M$. Then the element $\sum_{i=1}^{n} r_i x_i$ would obviously be went to $\sum_{i=1}^{n} r_i \phi(x_i)$. It should be fairly obvious that this is a module morphism (i.e., respect addition and scalar multiplication).

A special case of this is $R = \mathbb{Z}$, in which case we get the **free abelian group**.

Example A.2.3 (Free Group)**.**

Given a set X, the **free group** $F(X)$ is constructed as follows: an element of $F(X)$ is a words of letters x and x^{-1} for some $x \in X$. A word is reduced if there is no consecutive xx^{-1} or $x^{-1}x$ (because we can collapse it to identity). Each element of $F(X)$ is a reduced word.

The group operation is by concatenating two words and removing xx^{-1} or $x^{-1}x$, if exists. One can check that this satisfies the universal property.

§A.2.2 Products and Coproducts

We now define products and coproducts. The definition of product is a direct generalization of product of two objects as above.

Definition A.2.4.

Let $\mathscr{C}$ be a category, and let $(X_i)_{i\in I}$ be a collection of objects on $\mathscr{C}$. The **product** $\prod_{i\in I} X_i$ is the object X together with projection maps $\pi_i : X \to X_i$ that satisfies the following universal property:

for any object Y and morphism $f_i : Y \to X_i$, there exist a unique morphism $f : Y \to X$ such that $f_i = \pi_i \circ f$.

Example A.2.5.

Product in the category of sets, groups, abelian groups, and R-modules are simply the set of ordered pairs. In the case of groups, abelian groups, and R-modules, the corresponding operations are done coordinate-wise.

In general, whenever you see the word "co", it means that the definition is the same, except that all arrows are flipped. For example, the definition of coproduct is obtained by fliping the directions of all maps in the definition of products.

Definition A.2.6.

Let $\mathscr{C}$ be a category, and let $(X_i)_{i\in I}$ be a collection of objects on $\mathscr{C}$. The **coproduct** $\coprod_{i\in I} X_i$ is the object X together with inclusion maps $\iota_i : X_i \to X$ that satisfies the following universal property:

for any object Y and morphism $f_i : X_i \to Y$, there exist a unique morphism $f : X \to Y$ such that $f_i = f \circ \iota_i$.

Example A.2.7.

- Coproduct of sets is a disjoint union of two sets.
- Coproduct of R-modules is also known as direct sums: given collection of R-modules $(M_i)_{i\in I}$, each element of the coproduct $\coprod_{i\in I} M_i$ is a linear combination of finitely many elements in the M_i's.
- Coproduct of groups is known as free product. Its construction is explained in Section 2.3.1.

§A.2.3 Limit and Colimits

This section is not needed until Chapter 8, and even then it plays only a supplementary role.

In the definition of products and coproducts, we have a collection of objects that do not interact with each other. If those collection of objects have maps to each other, then the notion of product generalizes to limits, while the notion of coproduct generalizes to colimits.

Definition A.2.8 (Diagram)**.**

Let $\mathscr{C}$ be a category and let $\mathcal{I}$ be a category whose objects are sets (called a **small category**). A **diagram** is a functor from $\mathcal{I} \to \mathscr{C}$.

Most commonly, $\mathcal{I}$ will be a poset. In particular, for any poset $(\mathcal{I}, \leq)$, we can promote $\mathcal{I}$ to a category by including exactly one morphism from $i \to j$ if and only if $i \leq j$. A functor from $\mathcal{I} \to \mathscr{C}$ is a collection of objects indexed with $\mathcal{I}$, and a morphism for each object X_i, X_j with $i \leq j$.

Definition A.2.9.

Given a diagram $F : \mathcal{I} \to \mathscr{C}$ (sending i to F_i), the **limit** of this diagram is an object $X \in \mathscr{C}$ and a morphism $\pi_i : X \to F_i$ for each $i \in \mathcal{I}$ such that

- $F(\phi) \circ \pi_i = \pi_j$ for all morphisms $\phi : i \to j$
- for each collection of morphism $f_i : Y \to F_i$ such that $F(\phi) \circ f_i = f_j$ for all morphisms $\phi : i \to j$, there exists unique morphism $f : Y \to X$ such that $f = \pi_i \circ f$.

One can define colimit similarly by reversing all arrows.

The most important case at present is colimit of modules. In some sense, colimit provides a way to break down modules to finitely-generated ones. This is illustrated by the following examples.

Example A.2.10.

In the category of $\mathbb{Z}$-modules.

- Let $\mathcal{I} = \mathbb{N}$ with relation being the usual $\leq$. Consider the diagram M_i where $M_i = \mathbb{Z}$ and the map $M_i \to M_{i+1}$ is multiplication by 2. Then $\operatorname{colim} M_i = \mathbb{Z}[1/2]$.
- Let $\mathcal{I} = \mathbb{N}$ but with relation being divisibility (i.e., $i \prec j$ iff $i \mid j$). Consider the diagram M_i where $M_i = \mathbb{Z}$ and the map $M_i \to M_j$ is multiplication by j/i. Then $\operatorname{colim} M_i = \mathbb{Q}$.

We now present some properties of colimits.

Theorem A.2.11.

Colimit exists for all functor $F : \mathcal{I} \to R\text{-}\mathbf{Mod}$.

▶ *Proof.* See [AK13, Thm. 6.8] □

A key property that we will use is colimit preserves exactness.

Definition A.2.12.

A small category $\mathcal{I}$ is **filtered** if and only if

- given objects $i, j \in \mathcal{I}$, there exists a morphism $k \to i$ and $k \to j$;
- given two morphisms $f_1, f_2 : i \to j$, there exists object k and a map $g : j \to k$ such that $g \circ f_1 = g \circ f_2$.

Theorem A.2.13.

Given diagram $L, M, N : \mathcal{I} \to R\text{-}\mathbf{Mod}$ and morphisms between diagrams such that $L_i \to M_i \to N_i$ is exact for all $i \in \mathcal{I}$ and for all $\phi : i \to j$, the diagram

$$\begin{tikzcd} L_i \arrow[r] \arrow[d, "L(\phi)"] & M_i \arrow[r] \arrow[d, "M(\phi)"] & N_i \arrow[d, "N(\phi)"] \\ L_j \arrow[r] & M_j \arrow[r] & N_i \end{tikzcd}$$

commutes then the colimit

$$\operatorname{colim} L_i \to \operatorname{colim} M_i \to \operatorname{colim} N_i$$

is exact.

▶ *Proof.* See [AK13, Thm. 7.9] □

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