Gluing Genus 1 and Genus 2 Curves along ℓ-torsion

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Curves

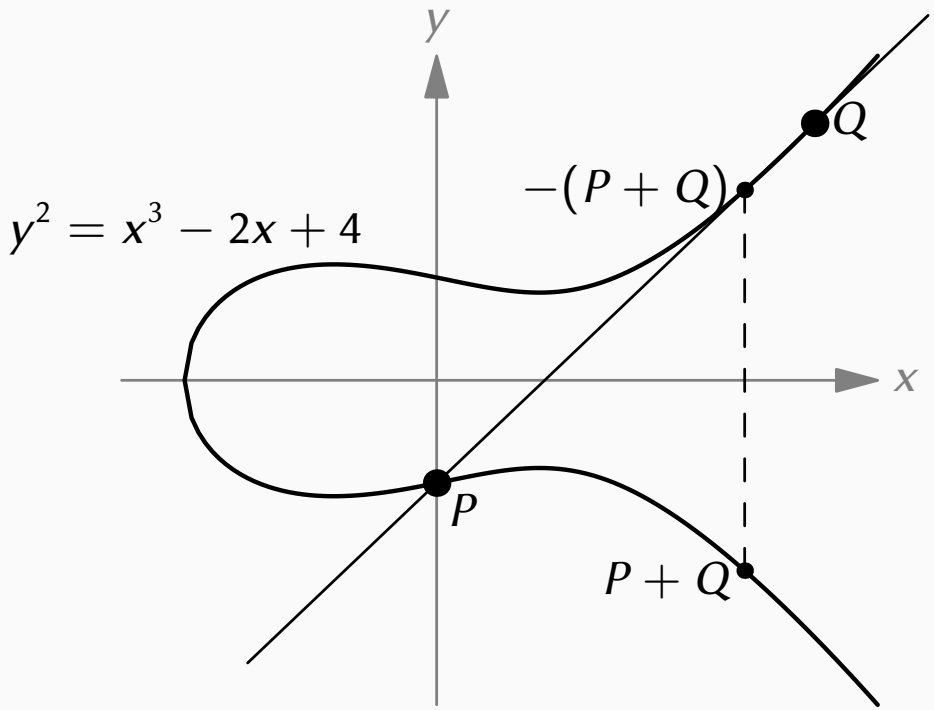
An (algebraic) **curve** is a system of polynomial equations whose solution set has dimension 1.

Viewed in complex projective space, it is a compact Riemann surface. The **genus** of a curve is the number of holes of that surface.

Example: $x^2 + y^2 = 1$ has genus 0.

Group law of Genus 1 Curves

Genus 1 curves (with a rational point) are known as **elliptic curves** (of the form $y^2 = x^3 + ax + b$). They have a group law.

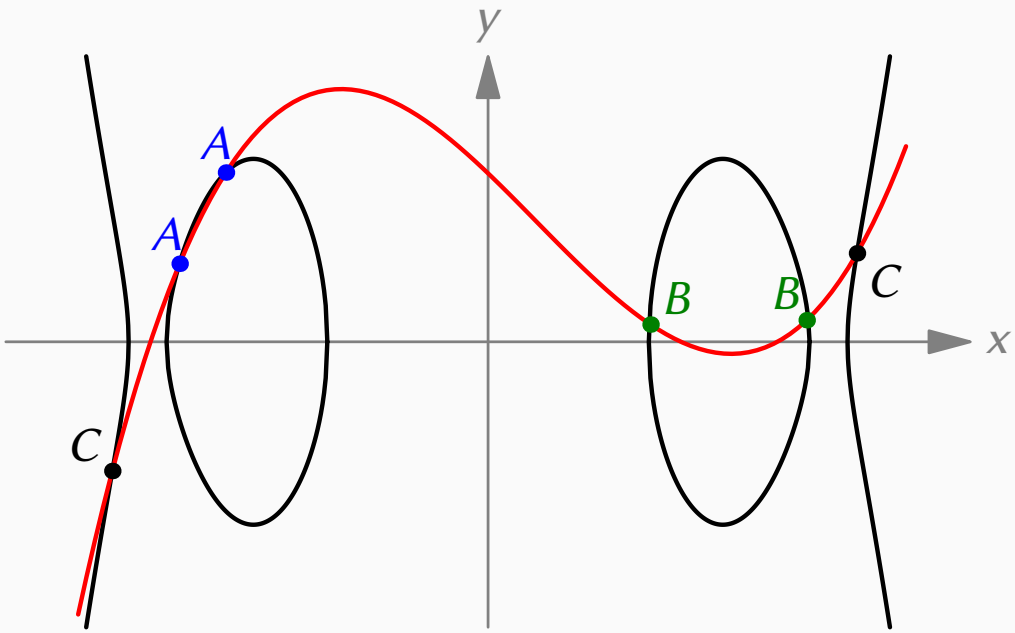


- P, Q, R lie on a line $\iff$ $P + Q + R = 0$.
- 0 is the point at infinity along y-axis.
- Big surprise: associative 😱‼

Genus 2 Curves

Genus 2 curves are of the form $y^2 = f(x)$ where $\deg f \in \{5, 6\}$.

They have group law on pairs of points.



$A + B + C = 0 \iff$ there is a cubic passing through A, B, C.

In general, for a genus g curve X, the **Jacobian** $\mathrm{Jac}\, X$ is both a group and a variety whose points parametrize g-tuple of points in X.

Gluing 📝

Let X and Y be curves over $\mathbb{Q}$.

A **gluing** of X and Y consists of a curve Z and a surjective map

$$\operatorname{Jac} X \times \operatorname{Jac} Y \twoheadrightarrow \operatorname{Jac} Z$$

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If $G \subseteq \operatorname{Jac}(X)[\ell] \times \operatorname{Jac}(Y)[\ell]$, then we say that the gluing is **along ℓ-torsion**.

We are interested in the case where **X is an elliptic curve**, **Y has genus 2**, and **ℓ is prime**, in which case the resulting gluing will have genus 3 (i.e., $1 + 2 = 3$).

Our Results 📈

We will discuss two results.

1. Given a genus 2 curve Y, we describe a framework to search for ℓ and X for which we can produce a gluing over $\mathbb{Q}$.
2. We describe a improved algorithm to compute gluing Z.

These allow us to find X and Y and glue them along 13-torsion. 😎

Gluability Criteria 🤔

👆 Fact.

$\mathrm{Jac}(X)[\ell] \simeq \mathbb{F}_\ell^2$ and $\mathrm{Jac}(Y)[\ell] \simeq \mathbb{F}_\ell^4$. Both are equipped with the bilinear alternating **Weil pairing**.

Not all subgroups $G \subset \mathrm{Jac}(X)[\ell] \times \mathrm{Jac}(Y)[\ell]$ lead to a gluing because the quotient must respect the Weil pairing.

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Hanselman-Schiavone-Sijsling: this means that G must be a **maximal isotropic subgroup**, which is parametrized by

- a one-dimensional subgroup $H \subset \mathrm{Jac}(Y)[\ell]$
- an antisymplectic isomorphism $\mathrm{Jac}(X)[\ell] \to H^\perp / H$.

There are $\approx \ell^6$ such subgroups, each of them produce a gluing over $\overline{\mathbb{Q}}$.

Gluability Criteria 🤔

It's very rare to get a gluing over $\mathbb{Q}$.

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- a **Gal($\overline{\mathbb{Q}}/\mathbb{Q}$)-stable** one-dimensional subgroup $H \subset \mathrm{Jac}(Y)[\ell]$
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Our 3-step workflow.

1. Find ℓ for which there exists a one-dimensional $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-stable subgroup $H \subset \mathrm{Jac}(Y)[\ell]$.
2. Find X such that $\mathrm{Jac}(X)[\ell] \simeq H^{\perp}/H$ (as $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-module).
3. Check that the isomorphism in (2) is antisymplectic.

Step 1. Finding Primes ℓ.

Running Example

We will demonstrate our workflow with Y being the curve `277.a.277.1` in the LMFDB:

$$y^2 + (x^3 + x^2 + x + 1)y = -x^2 - x,$$

which has conductor $N_Y = 277$.

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Condition 1.

There exists a one-dimensional $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-stable subgroup $H \subset \mathrm{Jac}(Y)[\ell]$.

This condition already restricts possible ℓ's to a finite set 😮.

Using Dieulefait's criterion, we can rule down to $\boxed{\ell \in \{3, 5\}}$.

Let's assume that we are looking for **gluing along 5-torsion** for now 👈.

Frobenius Polynomial 💡

For any prime p at which Y has a good reduction, there exists **Frobenius element** $\mathrm{Frob}_p \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ that

- acts like $(x : y : z) \mapsto (x^p : y^p : z^p) \pmod{p}$;
- acts on $\mathrm{Jac}(Y)[\ell] \simeq \mathbb{F}_\ell^4$ with a degree 4 characteristic polynomial congruent mod ℓ to the **Frobenius Polynomial** $F_{Y,p}(T) \in \mathbb{Z}[T]$, independent of ℓ.

Frobenius polynomials are easy to compute.

Step 2. Finding Elliptic Curves X 📌

Condition 2

$\mathrm{Jac}(X)[\ell] \simeq H^{\perp}/H$ (as $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-module).

We first understand H better by considering how Frob_p acts on H.

- Frob_p acts on H by multiplying by some number $\chi(p) \in \mathbb{F}_\ell^\times$, which must be a root of $F_{Y,p}(T)$ modulo ℓ.
- χ must be a Dirichlet character with conductor dividing 277 .

Out of all characters with conductor dividing 277, we eliminate ones which $\chi(p)$ is not a root of $F_{Y,p}(T)$ (modulo ℓ). Repeat this for several p's.

Eventually, we are left with $\chi = \chi_{\mathrm{triv}}$. In particular, Frob_p fixes H for any prime p.

(In general, χ may be nontrivial, but this process will very likely result in only one candidate χ corresponding to each H.)

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$\mathrm{Jac}(X)[\ell] \simeq H^{\perp}/H$ (as $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-module).

Note that Frob_p acts on

- H by multiplication by $\chi(p)$;
- $\mathrm{Jac}(Y)[\ell]/H^{\perp}$ by multiplication by $p/\chi(p)$.

The remaining two eigenvalues must come from $H^{\perp}/H$.

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Example

In our curve and $p = 19$:

$$\begin{aligned} F_{Y,19}(T) &= x^4 + x^3 - 22x^2 + 19x + 361 \\ &\equiv (x-1)(x-4)(x^2 - x + 4) \pmod 5. \end{aligned}$$

Thus, we know that **the trace of Frob_{19} in $\mathrm{Jac}(X)[\ell]$ is 1 mod 5**.

Step 2. Finding Elliptic Curves X 📌

Condition 2

$\mathrm{Jac}(X)[\ell] \simeq H^{\perp}/H$ (as $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-module).

For all p, we know what the trace of Frob_p in X must be (modulo ℓ).

Out of all elliptic curves in the LMFDB, there are 4 curves that satisfy our trace constraints for all $p \leq 100$:

`1939.b1` `18559.a1` `21883.b1` `32963.c1`

Step 3. Symplectic Test 📋

Condition 3

The isomorphism in condition 2 is anti-symplectic.

Freitas and Kraus: if Frob_p acts on $\mathrm{Jac}(X)[\ell]$ by a matrix conjugate to

$$\begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix},$$

then we can use this Frob_p to check whether the isomorphism is anti-symplectic.

We adapt this test and run it on our four candidates.

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All of them produce a gluing ✅.

Elliptic Curve	Resulting Gluing
`1939.b1`	$88189x^4 - 398531x^3y + 7700x^3z - 678120x^2y^2 + 1444780x^2yz + 231034x^2z^2 + 238603xy^3 - 1620885xy^2z = 0$
`21883.b1`	$y^2 = 448x^8 + 3584x^7 + 2016x^6 - 476x^5 - 13020x^4 - 16408x^3 - 18340x^2 - 8988x - 4025$
`32963.c1`	$19351616x^4 + 136748535x^3y + 106394158x^3z - 235515177x^2y^2 - 46043175x^2yz + 67674485x^2z^2 - 549641282xy^3 + 36999650xy^2z - 160500711xyz^2 - 36439076xz^3 + 272167382y^4 + 488584945y^3z - 488728851y^2z^2 + 152950443yz^3 - 115190535z^4 = 0$

Fast Gluing ⏩

Original algorithm (by Hanselman, Schiavone, Sijsling):

- Enumerate all $\approx \ell^6$ subgroups G.
- For each subgroup, compute the quotient $(\mathrm{Jac}(X) \times \mathrm{Jac}(Y))/G$ as a lattice in $\mathbb{C}^6$.
- Reconstruct the curve and check if it is defined over $\mathbb{Q}$.

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But we can improve! 😄

- For each one-dimensional subgroup $H \subseteq \mathrm{Jac}(Y)[\ell]$, check if it is $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-stable
  - by computing coordinates in $\mathbb{C}$ and rational-recognize its minimal polynomial.
- Enumerate all antisymplectic isomorphisms $\phi : \mathrm{Jac}(X)[\ell] \xrightarrow{\sim} H^\perp/H$. Then, use it to construct G.

$\approx \ell^3$ check in each step, so profit! 🎉

Timing

(Not including curve reconstruction)

	$\ell = 3$	$\ell = 5$	$\ell = 7$
Original	144.5 s	5760 s	(not attempted)
Fast	11.8 s	71.9 s	241 s

13-Gluing 😮

The current list of genus 2 curves in the LMFDB is not enough to produce (interesting) gluing along 13-torsion.

We use Sutherland's database and found the following 🔥:

$$X: \quad y^2 + y = x^3 + x^2 - 208x - 1256 \qquad \texttt{75.a1}$$
$$Y: \quad y^2 + x^3 y = -5x^4 + 45x^2 + 9x.$$

(Y has conductor 151 875.)

A gluing along 13-torsion of X and Y is

$$Z: y^2 = 1008x^8 - 4032x^7 + 336x^6 + 8064x^5 + 9660x^4 - 4914x^3 - 7434x^2 - 2478x + 2058.$$

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Questions ?