On t-intersecting Families of Permutations

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Intersecting Families of Permutations

Consider a collection $\mathcal{F}$ of permutations of $[n] := \{1, 2, \ldots, n\}$.

We say that $\mathcal{F}$ is ***t*-intersecting** if any two permutations in $\mathcal{F}$ agree at $\geq t$ inputs, i.e., for any $\sigma_1, \sigma_2 \in \mathcal{F}$, there exist $\geq t$ indices $i \in [n]$ such that $\sigma_1(i) = \sigma_2(i)$.

Question

For each $t < n$, what is the maximum size of t-intersecting family of permutations of $[n]$?

Constructions

Construction 1

The family

$$\mathcal{A}_0 = \{\sigma : \sigma(i) = i \text{ for all } i \in \{1, 2, \ldots, t\}\}$$

is clearly t-intersecting and has size $(n-t)!$.

For example, if $n = 7$ and $t = 4$, the elements in $\mathcal{A}_0$ are (in one-line notation)

$$\begin{array}{cc} (1,2,3,4,\mathbf{5},\mathbf{6},\mathbf{7}) & (1,2,3,4,\mathbf{5},\mathbf{7},\mathbf{6}) \\ (1,2,3,4,\mathbf{6},\mathbf{7},\mathbf{5}) & (1,2,3,4,\mathbf{6},\mathbf{5},\mathbf{7}) \\ (1,2,3,4,\mathbf{7},\mathbf{5},\mathbf{6}) & (1,2,3,4,\mathbf{7},\mathbf{6},\mathbf{5}) \end{array}.$$

This family turns out to be optimal when t is not too close to n.

Constructions

Construction 2

The family

$$\mathcal{A}_k = \left\{\sigma : \sigma(i) = i \text{ for } \geq t + k \text{ indices } i \text{ in } \{1, 2, \ldots, t + 2k\}\right\}$$

is clearly t-intersecting.

For example, if $n = 7$, $t = 4$, and $k = 1$, the elements in $\mathcal{A}_1$ are (in one-line notation)

$$\begin{array}{cc} \multicolumn{2}{c}{(1,2,3,4,5,6,\mathbf{7})} \\ (\mathbf{7},2,3,4,5,6,\mathbf{1}) & (1,\mathbf{7},3,4,5,6,\mathbf{2}) \\ (1,2,\mathbf{7},4,5,6,\mathbf{3}) & (1,2,3,\mathbf{7},5,6,\mathbf{4}) \\ (1,2,3,4,\mathbf{7},6,\mathbf{5}) & (1,2,3,4,5,\mathbf{7},\mathbf{6}) \end{array},$$

so $|\mathcal{A}_1| = 7$, larger than $|\mathcal{A}_0| = 6$.

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Conjecture (Deza and Frankl; Cameron)

Any t-intersecting family of permutations has size at most $\max_k |\mathcal{A}_k|$.

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This conjecture holds for the following ranges of n and t:

Range	Year	Authors	Method
$t \geq n - \omega(1)$	1977	Deza, Frankl	Elementary
$t = 1$	2003	Cameron, Ku	Elementary
$t \leq O(\log \log n)$	2011	Ellis, Friedgut, Pilpel	Rep. theory
$t \leq O\left(\frac{\log n}{\log \log n}\right)$	2022	Ellis, Lifshitz	Fourier + Rep. theory
$t \leq O\left(\frac{n}{\log^2 n}\right)$	2024	Kupavskii, Zakharov	Spread Approx.
$t \leq O(n)$	2024	Keller, Lifshitz, Minzer, Sheinfield	Fourier
$t \leq (1 - o(1))n$	2024+	Kupavskii	Spread Approx.

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Theorem (S., 2025+)

This conjecture holds for $t \leq n - O(n^{3/4+\varepsilon})$ for any $\varepsilon > 0$.

Partial Permutations

In the proof, it's more helpful to view a permutation as a subset of $[n]^2$. Permutation σ corresponds to the set

$$\{(1, \sigma(1)), (2, \sigma(2)), \ldots, (n, \sigma(n))\}.$$

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A **partial permutation** is a subset of $[n]^2$ that is contained in some permutation. An example of a partial permutation is

$$\{(1, 4), (3, 2), (6, 7)\}.$$

The **length** of a partial permutation is the size of the set (so the example above has length 3).

Kupavskii's Proof for $t \le (1 - o(1))n$

Let $\mathcal{F}$ be a t-intersecting family of permutations.

1. (Spread Approximation) Construct a set of partial permutations $\mathcal{S}$ such that
   - each element of $\mathcal{S}$ has length $\le t + 0.001(n - t)$.
   - $\mathcal{S}$ is t-intersecting: any two elements have intersection of size $\ge t$.
   - $\ge 99.9\%$ of elements in $\mathcal{F}$ contains at least one element of $\mathcal{S}$.

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2. (Peeling) Keep removing entries from a partial permutation in $\mathcal{S}$ while as long as $\mathcal{S}$ is still t-intersecting.

   In the resulting set, let $\mathcal{W}_k$ be the set of partial permutations of length $t+k$. Choose k such that $|\mathcal{W}_k| \geq t^{-0.5}\binom{t}{k}$ (i.e., large).

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3. (Counting) There exists $A, B \in \mathcal{W}_k$ such that $|A \cap B| = t$. Show that 99% of elements in $\mathcal{F}$ contains an element in $\binom{A \cup B}{t+k}$.

   Notice that $|A \cup B| = t + 2k$, so we get a stability result.

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   Notice that $|A \cup B| = t + 2k$, so we get a stability result.
4. (Finishing) Suppose that there exists an element not containing a set in $\binom{A \cup B}{t+k}$. Obtain a final contradiction by counting.

Our Improvement

Step 2, 3, and 4 can be easily modified to work if $t \leq n - O(n^{3/4+\varepsilon})$.

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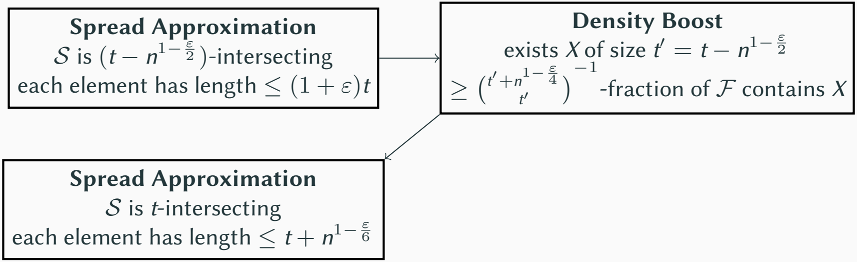
Kupavskii's proof works in the following way.



The key tool of the spread approximation step is the **sunflower lemma**, which is based on Alweiss-Lovett-Wu-Zhang's result and sharpened by Tao. This limits the bound that we can get.

Our Proof: Iterated Spread Approximation

Let $u = n - t$ and assume $u = \Omega(n^{0.6+\varepsilon})$.

Density Boost
exists X of length $t - 0.001u$
$\geq \left(\frac{u}{10n}\right)^{u}$-fraction of $\mathcal{F}$ contains X

→ **Spread Approximation**
$\mathcal{S}$ is $(t - 0.001u)$-intersecting
each elt. has length $\leq t + 0.998u$

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→ **Spread Approximation**
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$\vdots$

Density Boost
exists X of length $t - 0.001u$
$\geq \left(\frac{u}{10n}\right)^{0.003u}$-fraction of $\mathcal{F}$ contains X

→ **Spread Approximation**
$\mathcal{S}$ is $(t - 0.001u)$-intersecting
each elt. has length $\leq t + 0.001u$

Density Boost
exists X of length $t - 0.001u$
$\geq \left(\frac{u}{10n}\right)^{0.002u}$-fraction of $\mathcal{F}$ contains X

→ **Spread Approximation**
$\mathcal{S}$ is t-intersecting
each elt. has length $\leq t + 0.01u$

Thank you!

Towards the Complete Intersection Theorem for Permutations

The new Step 1 works for $t \leq n - O(n^{1/2+\varepsilon})$.

We now need to look at Step 3 (the counting step).

Original Proof

- Prove that there exists $A, B \in \mathcal{W}_k$ (with $t + k$ elements) such that $|A \cap B| = t$.
- Count the number of sets in $\mathcal{W}_k$ that intersect both A and B with size $\geq t$.
- Result: most elements in $\mathcal{W}_k$ are contained in $A \cup B$ (size $t + 2k$).

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Potential Improvement

- Prove that there exists $A_1, \ldots, A_r \in \mathcal{W}_k$ (with $t + k$ elements) such that [insert a condition]
- Count the number of sets in $\mathcal{W}_k$ that intersect each of $A_1, \ldots, A_r$ with size $\geq t$.
- Result: most elements in $\mathcal{W}_k$ are contained a set of size $t + 2k$.

Need to find a good condition to insert there.